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Parallel Numerics

Exercise 9: Domain Decomposition

1) Poisson Equation

Given is the Poisson problem

$$\begin{aligned} -\Delta u &= 1 \qquad u : \Omega = [0,1]^2 \mapsto \mathbb{R}, \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

with homogeneous Dirichlet boundary conditions.

i) Derive the following 5–point stencil for this problem using Finite Differences for equidistant grids:

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

ii) Set up the resulting matrix A_4 for a 4×4 grid. Enumerate the unknowns in ascending order.

iii) Set up the resulting matrix A_9 for a 9×9 grid. Enumerate the unknowns in ascending order.

iv) Set up the resulting matrix A_9 for a 9×9 grid. Enumerate the unknowns according to the red–black GS.

2) Domain Decomposition I: Dissection

i) Reenumerate the unknowns of the 9×9 grid: First enumerate the unknowns within $\Omega_1 = [1..4] \times [1..4]$, then the unknowns within $\Omega_2 = [6..9] \times [1..4]$, $\Omega_3 = [1..4] \times [6..9]$ and $\Omega_4 = [6..9] \times [6..9]$. Finally, assign a number to all the unknowns left (Ω_5).

ii) Set up the global stiffness matrix $\hat{A}_9$ according to the new enumeration. Write $\hat{A}_9$ using the matrix A_4.

iii) Assume all unknowns except the unknowns within Ω_1 are known. How does the equation system for Ω_1 look like?

iv) Assume all unknowns within $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 are known. How does the equation system for Ω_5 look like?

v) Define a parallel algorithm in pseudocode (own words) using the two small equation systems derived before.

3) Domain Decomposition II: A Non–overlapping Method

To set up matrix A, one has to set up one equation per unknown. Instead of traversing all the unknowns of the grid, one could also examine every cell and accumulate the whole system using the element–wise operator

$$\frac{1}{h^2}\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

i) Assume the unknowns within the 9×9 grid are enumerated in ascending order. How is the 14th line of matrix A_9 set up step by step?

ii) Give a Jacobi-wise processing scheme using the element–wise matrices. Focus on a subsection of the whole grid (e.g. scheme for vertex 14) and use the residual formulation. Note, that there is no formula requested, just a processing scheme.

iii) Assume the grid is split up into two sub domains $[1..5] \times [1..9]$ and $[5..9] \times [1..9]$. Derive a parallel Jacobi-wise processing scheme without setting up the stiffness matrix explicitly.

4) The Quality of a Partition

From a parallelisation point of view, a partition is "optimal" if it has a small boundary, a big volume and few neighbouring partitions.

i) Give reasons for this statement.

ii) What would be an optimal partition for the 9×9 grid.

iii) Outlook: In the lecture "Algorithmen des Wissenschaftlichen Rechnens" it is proven, that space–filling curves are Hölder continuous. Assume one splits up a computational domain according to the approximating polygon of a space–filling curve. What does this mean for the quality of the partition?