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Parallel Numerics

Exercise 9: Domain Decomposition

1 Poisson Equation

Given is the Poisson problem

$$\begin{aligned} -\Delta u &= 1 \qquad u : \Omega = [0,1]^2 \mapsto \mathbb{R}, \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

with homogeneous Dirichlet boundary conditions.

a) Derive the following 5-point stencil for this problem using Finite Differences for equidistant grids:

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

b) Set up the resulting matrix A_4 for a 4×4 grid (ignoring boundary conditions). Enumerate the unknowns in ascending order.

c) Set up the resulting matrix A_9 for a 9×9 grid (ignoring boundary conditions). Enumerate the unknowns in ascending order.

d) Set up the resulting matrix A_9 for a 9×9 grid (ignoring boundary conditions). Enumerate the unknowns according to the red-black GS.

2 Domain Decomposition I: Dissection

a) Reenumerate the unknowns of the 9×9 grid: First enumerate the unknowns within $\Omega_1 = [1..4] \times [1..4]$, then the unknowns within $\Omega_2 = [6..9] \times [1..4]$, $\Omega_3 = [1..4] \times [6..9]$ and $\Omega_4 = [6..9] \times [6..9]$. Finally, assign a number to all the unknowns left (Ω_5).

b) Set up the global stiffness matrix $\hat{A}_9$ according to the new enumeration (ignoring boundary conditions). Write $\hat{A}_9$ using the matrix A_4.

c) Assume all unknowns except the unknowns within Ω_1 are known. How does the equation system for Ω_1 look like?

d) Assume all unknowns within $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 are known. How does the equation system for Ω_5 look like?

e) Define a parallel algorithm in pseudocode (own words) using the two small equation systems derived before.

3 Domain Decomposition II: A Non-overlapping Method

To set up matrix A, one has to set up one equation per unknown. Instead of traversing all the unknowns of the grid, one could also examine every cell and accumulate the whole system using the element-wise operator

$$\frac{1}{h^2}\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

a) Assume the unknowns within the 9×9 grid are enumerated in ascending order. How is the 14th line of matrix A_9 set up step by step?

b) Give a Jacobi-wise processing scheme using the element-wise matrices. Focus on a subsection of the whole grid (e.g. scheme for vertex 14) and use the residual formulation. Note, that there is no formula requested, just a processing scheme.

c) Assume the grid is split up into two sub domains $[1..5] \times [1..9]$ and $[5..9] \times [1..9]$. Derive a parallel Jacobi-wise processing scheme without setting up the stiffness matrix explicitly.