Technische Universität München
Institut für Informatik
Lehrstuhl für numerische Programmierung

Introduction to Scientific Computing I

Final Exam – January, 31st, 2006

Name: ....
Matr.Nr.: ....
Course: ....

General Instructions

Material: You may only use one hand-written sheet of paper (size A4, on both pages). All other material including electronic devices of any kind are forbidden. You may use this exercises sheet and the exam paper that was handed out to solve the exercises (for notes and rough sketches, you can obtain additional exam sheets). Do not use pencil, or red or green ink.

Exercises: Nearly all exercises can be solved without the results from the previous exercises: if you are stuck with exercise a, then don't immediately skip exercises b, c, ...

Maximum score: The maximum score is 40 points plus 10 points for the bonus question. 17 points are required to pass the exam. The working time is 90 minutes.

1 Direction Fields for ODE (2 + 3 + 3 + 3 = 11 points)

Consider the direction field given in figure 1.

a) In the lectures, we examined an ODE population model that leads to such a direction field. Give the name of the model, and write down the general form of the ODE.

(You do not need to give explicit values for parameters. For example, if the ODE would be $f' = \alpha f^2 + \beta f^3$*, the values of* α *and* β *need not be given.)*

b) Give the formula for the explicit and the implicit Euler scheme for this ODE.

(The general formula of the scheme is sufficient – do not give explicit expressions for p_{n+1}*, such as* $p_{n+1} = \ldots$ *If you are not able to solve exercise a), then give the implicit/explicit Euler scheme for the ODE* $f' = \alpha f^2 + \beta f^3$*.)*

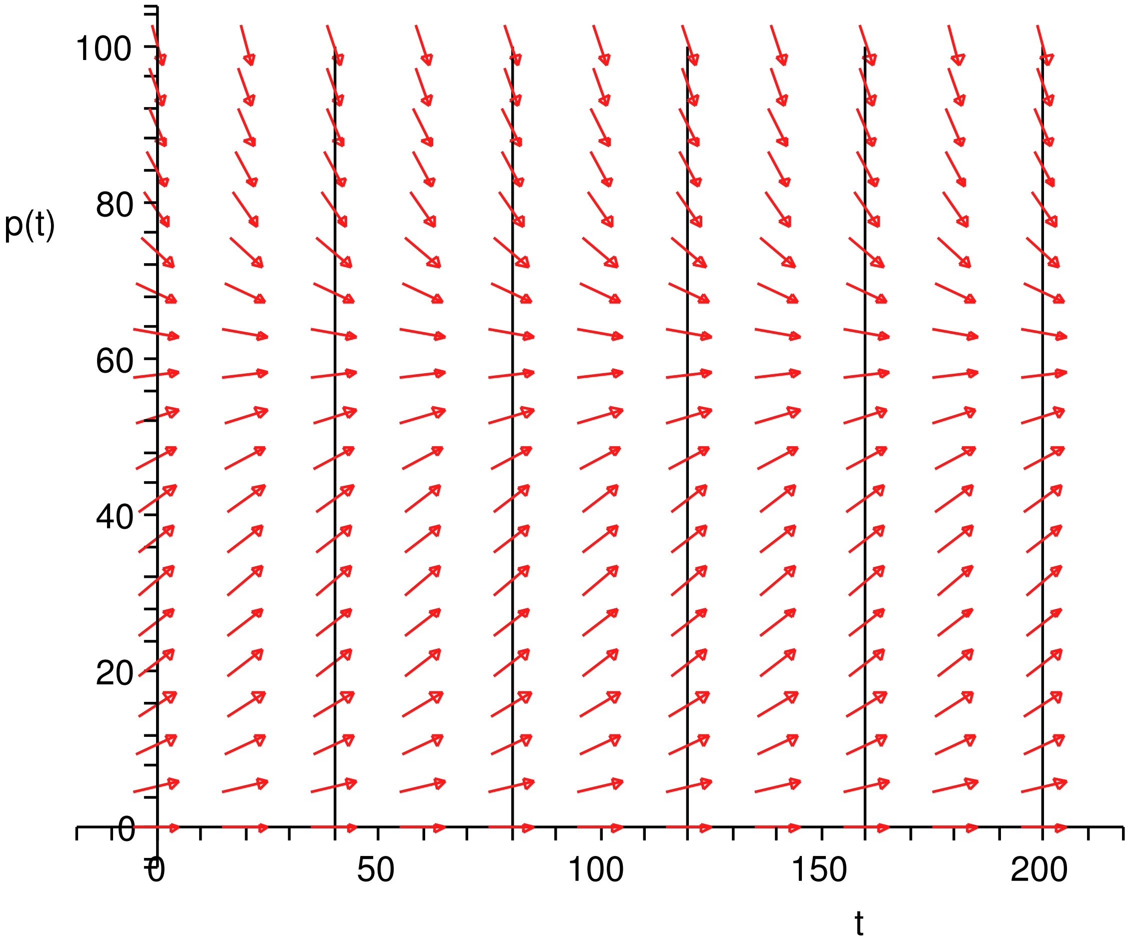

Figure 1: Direction field for exercise 1

c) The initial size of the modelled population shall be $p_0 = 10$. Perform the first 5 steps of the explicit Euler scheme by drawing the approximate solutions and the linear interpolation into the direction field in figure 1. The stepsize shall be defined by the 5 intervals drawn into the direction field. Mark from which arrows you obtain the directions of the Euler steps.

d) Now consider the Runge-Kutta type scheme defined by

$$p_{n+1} = p_n + \tau f\left(t_n + \frac{\tau}{2}, y_n + \frac{\tau}{2} f(t_n, y_n)\right)$$

For initial size of population $p_0 = 90$, perform the first 2 steps of this Runge-Kutta scheme using the same stepsize τ as for the Euler schemes. Again, draw the approximate solutions, their linear interpolation, and mark which arrows where involved for computing the directions.

2 Population Modelling (4 + 2 + 5 = 11 points)

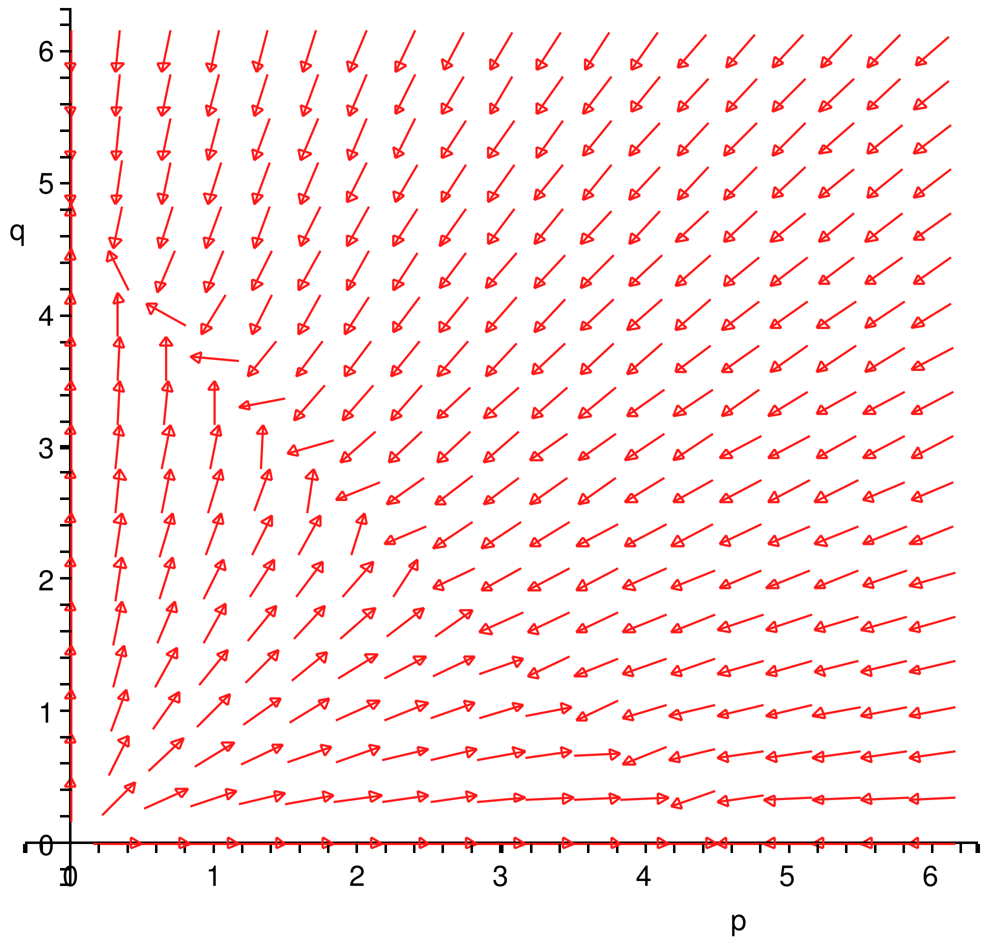

Figure 2: Direction field for exercise 2

Figure 2 shows the direction field for a system of ODE that models two populations $p(t)$ and $q(t)$.

a) Mark the critical points in the direction field (numbering them $1, 2, \ldots$), and state the type of critical point for each point you marked.

b) Does the direction field result from a linear or a non-linear model? Give reasons for your opinion.

c) Give a system of ODEs that could lead to this direction field. Give the signs of the parameters you used, and explain what populations scenario was probably modelled here.

(Again, you do not need to give explicit values for the parameters. For example, if the ODE would be $f' = \alpha f^2$, just state whether α should be positive or negative.)

3 Numerics for PDE (3 + 3 + 2 = 8 points)

Consider the 2D heat equation

$$\alpha \frac{\partial^2}{\partial x^2} u(x,y) + \beta \frac{\partial^2}{\partial y^2} u(x,y) = 0 \quad \text{on } \Omega = [0,1]^2, \tag{1}$$

where there are diffusion coefficients α and β for the diffusion in x- and y-direction. You may disregard the boundary conditions for the following exercises.

a) Assume that equation (1) has analytical solutions of the form $u(x,y) = v(x)w(y)$, and derive a set of two ordinary differential equations for the two functions $v(x)$ and $w(y)$.

b) Formulate a finite difference scheme to compute approximate solutions of equation (1).

c) What principle approaches are there to improve the accuracy of the numerical solution? (Name two of them.)

4 Boundary Conditions (2 + 3 = 5 points)

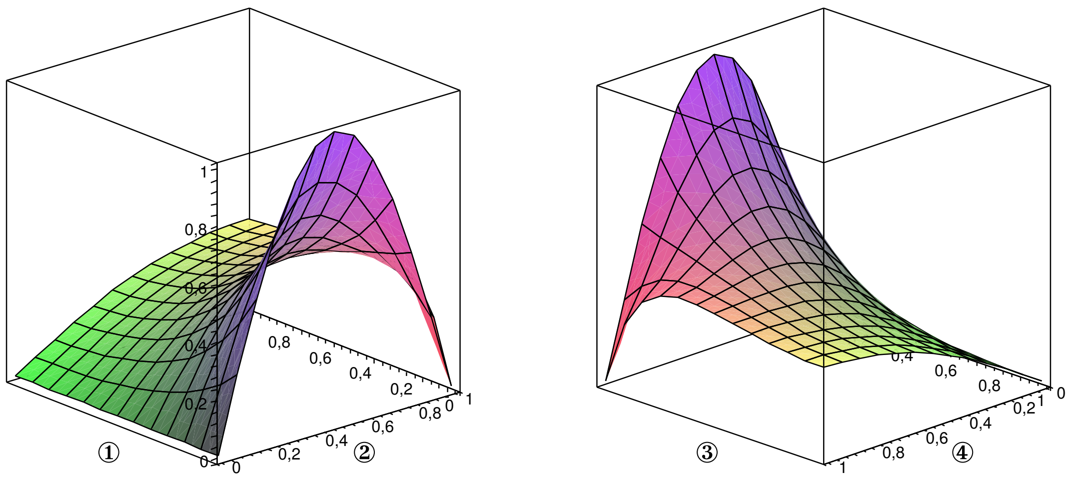

Figure 3: Temperature plot for exercise 4 – the same function is plotted from two different angles.

Figure 3 shows the solution (plot of the function u) of the heat equation (1) for the simple case $\alpha = \beta = 1$ on the domain $\Omega = [0,1]^2$.

(Note: both plots show the same function, but from different angles!)

a) What types of boundary conditions have been used at the boundaries ①, ②, ③, and ④? (Hint: a non-homogeneous boundary condition was used at only one of the boundaries.)

b) Give a numerical scheme to implement the boundary conditions at the boundaries ① and ④.

5 Grid Generation (3 + 2 = 5 points)

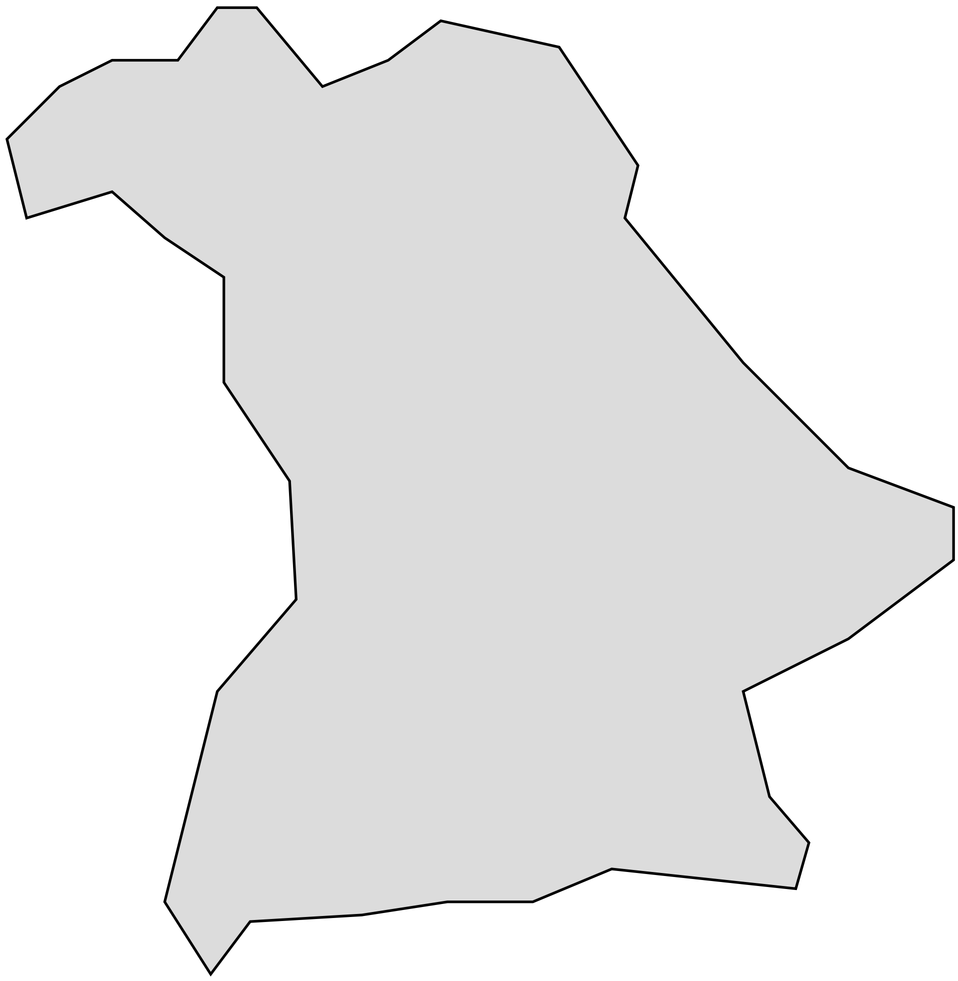

Figure 4: Computational domain for exercise 5.

Figure 4 shows a complicated domain.

a) Choose a grid generation approach that is suitable for this domain. Apply this grid generation approach, and draw a respective grid into figure 4.

(You only need to draw enough cells to make the principle clear.)

b) Classify your grid generation approach (keywords are sufficient).

Bonus Question: Finite Elements (2 + 3 + 5 = 10 points)

Consider the **2D**-problem

$$\frac{\partial^2}{\partial x^2} u(x,y) = f(x,y) \tag{2}$$

on a two-dimensional domain Ω with homgeneous Dirichlet boundary conditions.

a) Give a weak formulation of equation (2).

b) For a finite element discretization, consider a single element that consists of the triangle bounded by the three grid points $x_1 = (0,0)$, $x_2 = (0,h)$, and $x_3 = (h,0)$. If we use linear basis functions of the form $\varphi_j = ax + by + c$, then the basis functions corresponding to the grid points x_1 and x_2, respectively, are

$$\begin{aligned} \varphi_1(x,y) &= 1 - \tfrac{1}{h}x - \tfrac{1}{h}y \\ \text{and } \varphi_2(x,y) &= \tfrac{1}{h}x\,. \end{aligned}$$

Compute the basis function $\varphi_3(x,y)$ corresponding to x_3.

c) Compute two matrix elements of the element stiffness matrix A for the given triangular element, for example A_{12} and A_{22}.