Technische Universität München
Institut für Informatik
Lehrstuhl für numerische Programmierung

Scientific Computing I

Final Exam – January, 31st, 2007

Name: ....

Matr.Nr.: ....

Course: ....

General Instructions

Material: You may only use one hand-written sheet of paper (size A4, on both pages). All other material including electronic devices of any kind are forbidden.

You may use this exercise sheet and the exam paper that was handed out to solve the exercises (for notes and rough sketches, you can obtain additional exam sheets).

Do not use pencil, or red or green ink.

Exercises: Very often, exercises b), c), etc. can be solved without the results from the previous exercise a): if you are stuck with exercise a, then don't immediately skip exercises b, c, etc.

Maximum score: The maximum score is 40 points plus a bonus of 8 points. Grades will be computed relative to a maximum score of 40 points.

17 points are required to pass the exam.

Working time: 90 minutes.

1 Direction Fields for ODE (2 + 3 + 3 + 3 = 11 points)

Consider the direction field given in figure 1.

a) In the lectures, we examined an ODE population model that leads to such a direction field. Give the name of the model, and write down the general form of the ODE.

You do not need to give explicit values for parameters. For example, if the ODE would be $f' = \alpha f^2(1+\beta f)$*, the values of* α *and* β *need not be given.*

b) Give the formula for the explicit and the implicit Euler scheme for this ODE.

The general formula for the schemes is sufficient – do not solve for an explicit expressions for p_{n+1}*. If you are not able to solve exercise a), then give the explicit and implicit Euler scheme for the ODE* $f' = \alpha f^2(1+\beta f)$*, instead.*

c) The initial size of the modelled population shall be $p_0 = 38$. Perform the first four steps of the *implicit* Euler scheme by drawing the approximate solutions and the linear interpolation into the direction field in figure 1.
The stepsize shall be $\tau = 1$, as illustrated by the four intervals drawn into the direction field. Mark from which arrows you obtain the directions of the Euler steps – sometimes, you will need to add an arrow to the direction field, if no arrow is plotted at the required position.

d) Now consider the Runge-Kutta scheme defined by

$$p_{n+1} = p_n + \frac{\tau}{2}\Big(f(t_n, p_n) + f\big(t_{n+1}, p_n + \tau f(t_n, p_n)\big)\Big)$$

For initial size of population $p_0 = 2$, perform the first step of this Runge-Kutta scheme, now using the stepsize $\tau = 2$ (i.e. one timestep extends over two of the drawn intervals). Again, draw the approximate solutions, their linear interpolation, and mark which arrows where involved for computing the directions.

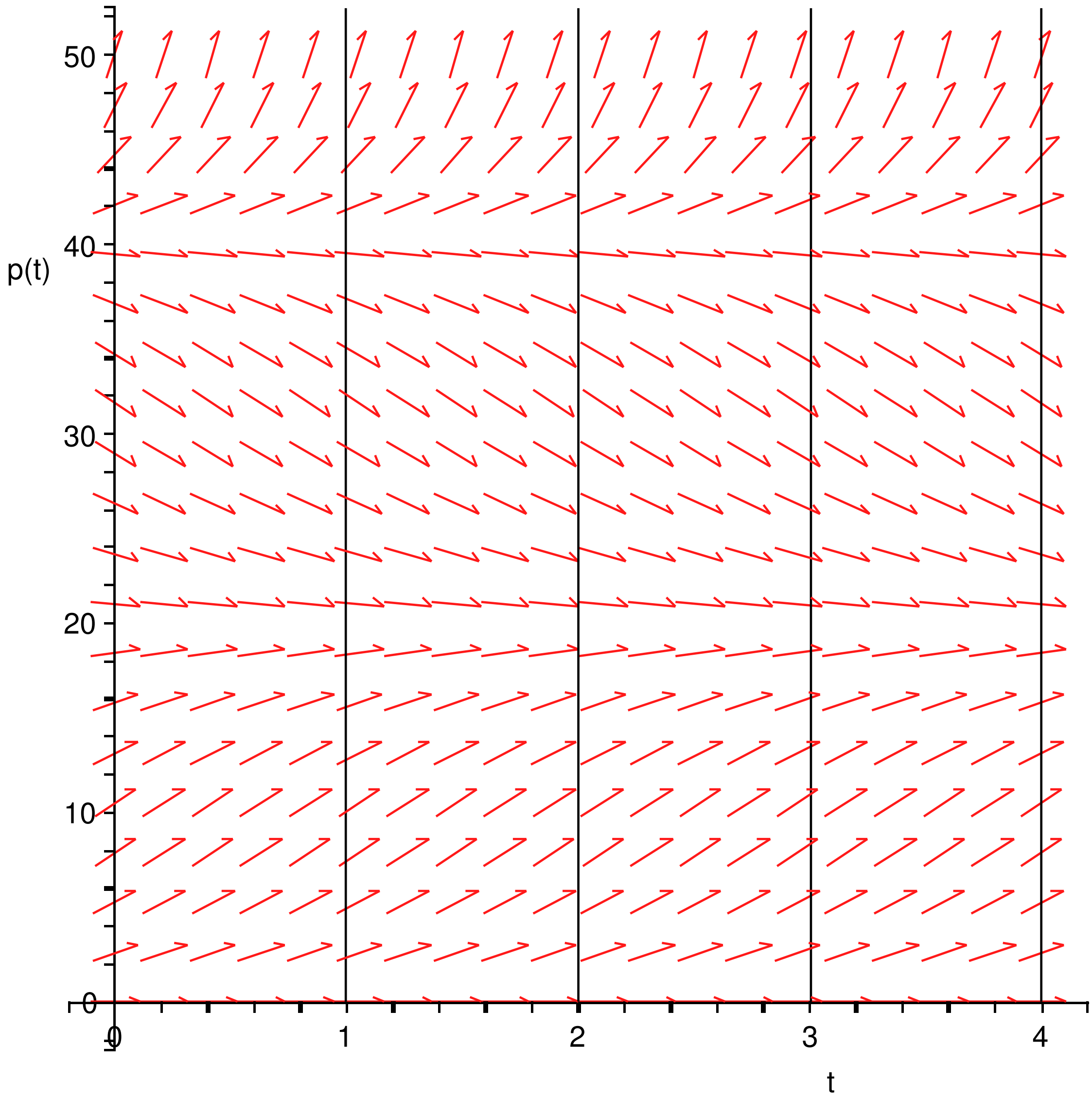

Figure 1: Direction field for exercise 1 – draw the solutions of exercises 1c and 1d directly into this diagram.

2 Population Modelling (4 + 2 + 5 = 11 points)

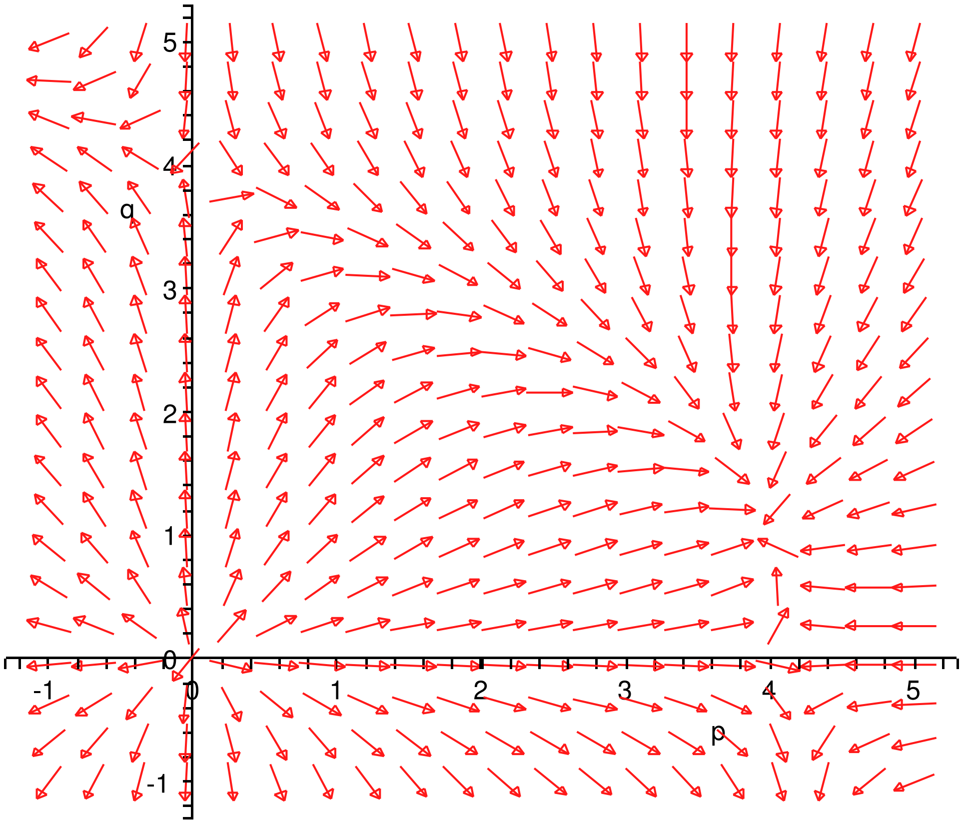

Figure 2: Direction field for exercise 2. Population $p(t)$ is plotted along the horizontal axis, population $q(t)$ is plotted along the vertical axis.

Figure 2 shows the direction field for a system of ordinary differential equations that model two populations $p(t)$ and $q(t)$.

a) Mark the critical points in the direction field (numbering them $1, 2, \ldots$), and state the type of critical point for each point you marked.

b) Does the direction field result from a linear or a non-linear model? Give reasons for your opinion.

c) Give a system of ODEs that could lead to this direction field. Give the signs of the parameters you used, and explain what populations scenario was probably modelled here.

(Again, you do not need to give explicit values for the parameters. For example, if the ODE would be $f' = \alpha f^2$, just state whether α should be positive or negative.)

3 Numerics for PDE, Neumann Stability (5 + 3 + 2 = 10 points)

The discretisation of the time-dependent 1D heat equation using the so-called *leap-frog* scheme (or midpoint rule) leads to the following numerical scheme:

$$\frac{u_j^{(m+1)} - u_j^{(m-1)}}{2\tau} = \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{h^2} \tag{1}$$

with Dirichlet boundary conditions: $u_0^{(m)} = u_n^{(m)} = 0$ for all m.

As usual, we denote $u_j^{(m)} \approx u(t_m, x_j)$, which means that in the approximate solution $u_j^{(m)}$ the superscript (m) denotes the time step, and the index j denotes the grid point.

a) Similar to the lectures, we assume that equation (1) has solutions of the form

$$u_j^{(m)} := (a_k)^m \sin(k\pi x_j)$$

for different values of k. Determine the respective value(s) of a_k.

Hint 1: *You will find two possible solutions for each* a_k

Hint 2: *You may use that* $x_{j\pm1} = x_j \pm h$, *and that* $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$.

If you cannot solve exercise a), then for exercises b) and c) you may assume that $(a_k)_{1/2} = \frac{1}{2}\left(-C_k \pm \sqrt{C_k^2 + 4}\right)$, where $C_k \geq 0$.

b) State what properties the a_k should have to obtain a sensible solution, and why this is required. Check whether these properties are satisfied. If the properties are only satisfied for certain values of τ and h, then specify these restrictions.

c) Is the leap-frog scheme numerically stable for the 1D heat equation that we studied here?

4 Finite Elements (2 + 2 + 6 = 10 points)

Consider the 1D-problem

$$\frac{\partial}{\partial x} u(x) = f(x) \tag{2}$$

on the unit interval $\Omega = (0, 1)$ with homgeneous Dirichlet boundary conditions: $u(0) = u(1) = 0$.

a) Give a weak formulation of equation (2).

b) Draw a picture of the nodal basis for the equidistant grid points $x_j := j/4$ for $j = 1, 2, 3$; why is it justified to have no basis functions for the grid points $x_0 := 0$ and $x_4 := 0$?

c) Compute the system matrix for a finite element discretization, where you use the nodal basis of b) for test and ansatz functions. You only need to compute one typical row of the matrix (for example, the row corresponding to the unknown at grid point x_2).

You may "compute" the value of integrals in a graphical way, just as we did in the lectures.

5 Discretisation Stencils (3 + 3 = 6 points)

Figure 3 shows a uniformly refined, regular grid for a Finite Element discretisation of the 2D Poisson equation. The unknowns u_{ij} (illustrated by the black dots in the grid) are placed in the standard equidistant way.

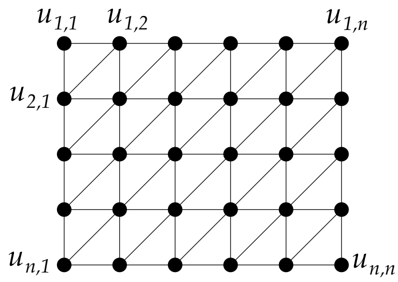

Figure 3: A uniformly refined regular grid with triangular grid cells – the unknowns are placed on the black dots.

You look up the respective FEM-discretisation for this problem in a textbook; the textbook provides the following 7-point discretisation stencil:

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

a) Write down one typical line of the system of linear equations that is described by this discretisation stencil (right hand sides shall be given by the values f_{ij}).

b) The system of linear equations in a) shall now be written in matrix-vector form: $Ax = f$, where

$$x = (u_{1,1}, \ldots, u_{1,n}, u_{2,1}, \ldots, u_{2,n}, \ldots, u_{n,1}, \ldots, u_{n,n})$$

is the vector containing the unknowns. Describe the system matrix A.