Technische Universität München
Institut für Informatik
Dr. habil. Miriam Mehl

WT 2011/12

Introduction to Scientific Computing

Final Exam Repetition, April 16th 2012

Name: ....

Matr.Nr.: ....

Programm: ....

General Instructions

Material: You may only use one hand-written sheet of paper (size A4, on both pages). All other materials including electronic devices of any kind are forbidden. You may use this exercise sheet and the exam paper that was handed out to solve the exercises (for notes and sketches, you can obtain additional exam sheets). Do not use a pencil or red or green ink.

General hint: Often, exercises b), c), etc. can be solved without the results from the previous exercise a): if you are stuck with exercise a, then do not immediately skip exercises b, c, etc.

Maximum score: The maximum score is 40 points. 17 points are required to pass the exam.

Working time: 90 minutes.

1 Population Modelling (2+4+3+1+2+2 pts)

We consider a given model for the population p of a given species

$$\dot{p}(t) = p(t)\left(2 - \frac{1}{2}p(t)\right). \quad (1)$$

a) Compute the critical points of this model.

b) Give the type of the critical points and sketch the direction field.

c) Determine the order of consistency of the explicit Euler method

$$p^{(n+1)} = p^{(n)} + dt \cdot p^{(n)} \left(2 - \frac{1}{2} p^{(n)}\right), \quad n = 1, 2, \ldots, \tag{2}$$

where $p^{(n)}$ denotes the approximation of $p(t_n)$ with $t_n = n \cdot dt, dt > 0$.

d) Apply the explicit Euler method for initial value $p(0) = 10$ with time step size $dt = \frac{1}{4}$. Perform one Euler step.

e) What's the limit $\lim_{n\to\infty} p^{(n)}$ in **d)**, what would be the limit $\lim_{t\to\infty} p(t)$ of the exact solution of (1) with $p(0) = 10$..

f) Can you conclude from the result of **e)**, that the explicit Euler method does not converge?

2 Numerics for ODEs (3 pts)

We solve an ordinary differential equation

$$\dot{p}(t) = f(t, p(t)) \text{ in }]0; T[, \quad (3)$$
$$p(0) = p_0 \quad (4)$$

numerically using the fourth order explicit Runge-Kutta method.

a) Write down the formulas for one time step for equation (3) with step size dt.

b) Is their any inherent parallelism in the RK method from **a)**? If yes, indicate which steps can be executed in parallel.

c) Give the name of a method that can be used for parallel time-stepping.

3 PDE Numerics (2+5+3+5 pts)

We solve the one-dimensional partial differential equation

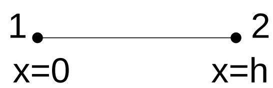
$$u_{xx}(x) + u(x) = f(x) \in \Omega, \tag{5}$$
$$u(x) = 0 \text{ at } \partial\Omega \tag{6}$$

on a domain Ω.

a) Give the weak form of (5).

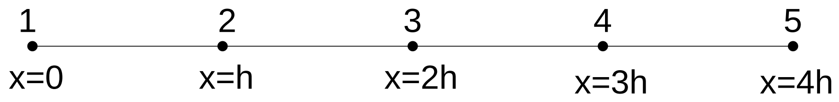
b) We solve (5) using finite elements on a regular grid. Compute the element stiffness matrix for the interval displayed below with the linear nodal basis functions

$$\phi_1(x) = 1 - \frac{x}{h}, \tag{7}$$
$$\phi_2(x) = \frac{x}{h}. \tag{8}$$



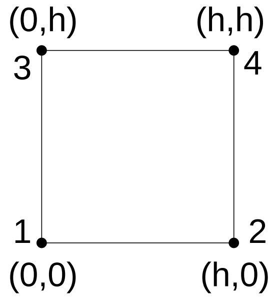
where ϕ_i is the basis function associated to node i of the interval. Test and ansatz space are the same in our example (both given by $\phi_i, i = 1, 2$ on the interval).

c) Use the results from **b)** to assemble the matrix for the following grid consisting of four elements:



d) Assume now, that we have to solve a two-dimensional partial differential equation on a regular Cartesian grid and that the element matrix for a square element displayed below is given as

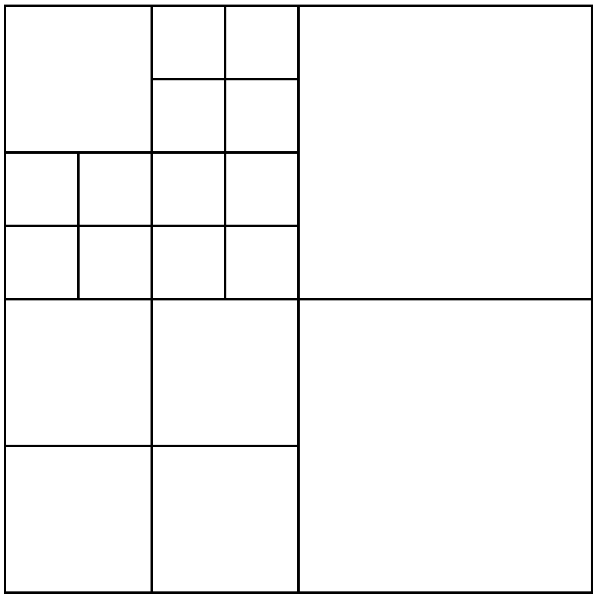
$$A_e = \begin{pmatrix} -2 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -2 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & -2 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & -2 \end{pmatrix} \tag{9}$$



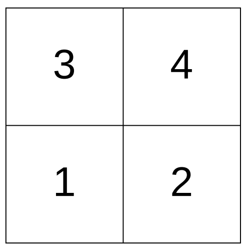
Derive the discretization stecil for a given grid point in a regular Cartesian grid.

4 Computational Grids (4 pts)

Assume, we consider the following adaptive Cartesian grid



We want to derive a linearized bit-code decribing the underlying cell tree. We know, that the cells of the grid, a two-dimensional quadtree grid, are ordered following the so-called Morton order (equal on all levels without rotations or mirroring):



Further, the ordering is done in a depth-first manner (descend to children before going on to the neighbour on the same level). Derive the respective linearized bit-code of the given grid (1 stands for a refined cell, 0 for a unrefined cell).