General Instructions

Material:

You may only use one hand-written sheet of paper (size A4, on both pages).

Any other material including electronic devices of any kind is forbidden.

Use only the exam paper that was handed out to solve the exercises. In case the space on a page is not enough, mark that you continue with your solution and use the reverse side of the preceding page. For additional notes and sketches, you can obtain additional exam sheets.

Do not use pencil, or red or green ink.

General hint:

Often, exercises b), c), etc. can be solved without the results from the previous exercise a); if you are stuck with exercise a), then don't immediately skip exercises b), c), etc.

Working time:

90 minutes + 5 minutes reading time.

Please switch off your cell phones!

1	2	3	4	$\sum$
/16	/9	/14	/9	/≈ 48

1 Particle in the Rotating Tube (≈ 1 + 1 + 5 + 4 + 5 = 16 points)

A small spherical particle is located in a very long rotating tube full of liquid, which does not move relative to the tubes walls.
We are interested in the particle motion along the tube axis x, see Figure 1. Therefore, the rotating

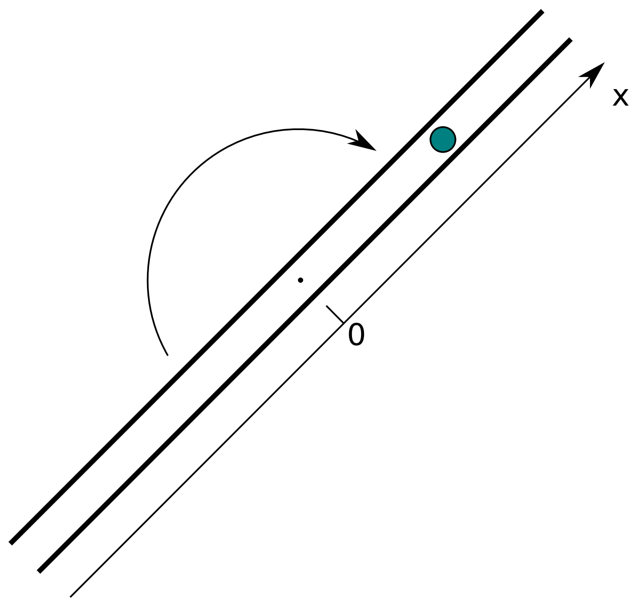

Figure 1: A rotating tube with liquid and particle inside.

reference frame is used to derive the equation of the particle motion. In this reference frame, we consider two dominating forces acting on the particle along x axis: the Stokes drag force and the buoyancy type force.
The model is approximated by the following ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = (1-\gamma)\,x - \lambda \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1}$$

where $\gamma = \rho_l/\rho_p > 0$ is the ratio of densities (ρ_l, ρ_p – liquid and particle densities) and $\lambda \geq 0$ is the friction coefficient.

(a) Transform the second order ODE (1) into a system of first-order ODEs.

(b) The matrix-vector form of the linear system of ODEs is given by

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ 1-\gamma & -\lambda \end{pmatrix} \mathbf{y}. \tag{2}$$

Find the critical point of the system.

(c) Study how the type of the fixed point depends on γ, when $\lambda = 1$. Name all possible types of critical points and γ ranges at which these types occur. For the critical point classification you may use Figure 2.

Last name, first name, student ID:

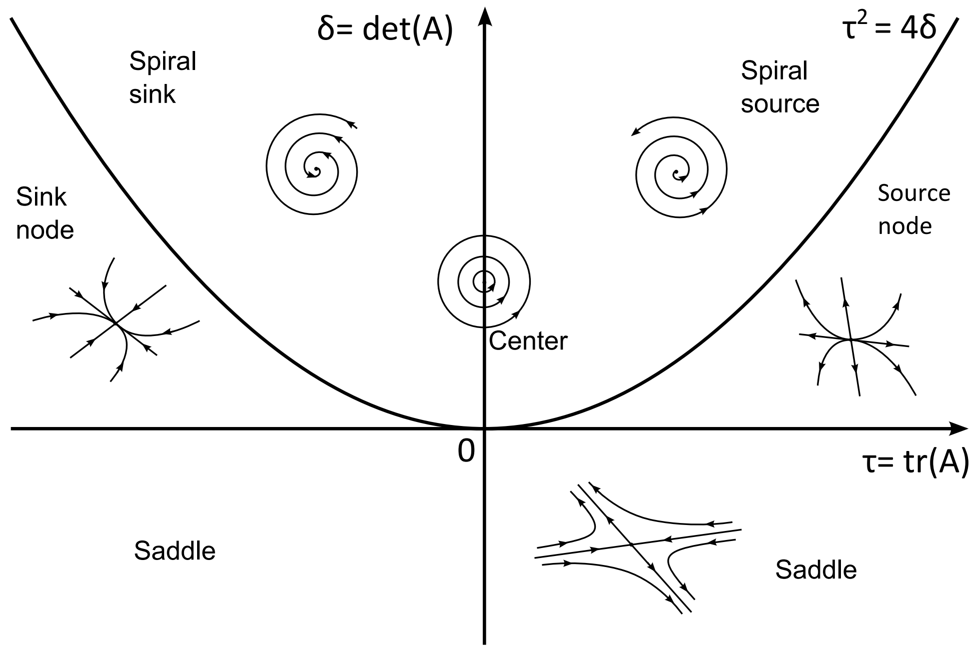

Figure 2: Type and stability of a critical point in two dimensions.

Last name, first name, student ID:

(d) Sketch the direction field of the system of ODEs with fixed $\gamma = 1/4$ and $\lambda = 1$ in Figure 3. Use eigenvalues and eigenvectors to determine the direction field.

Hint: use the right image in case you need to redraw your solution (cross out the first attempt in that case).

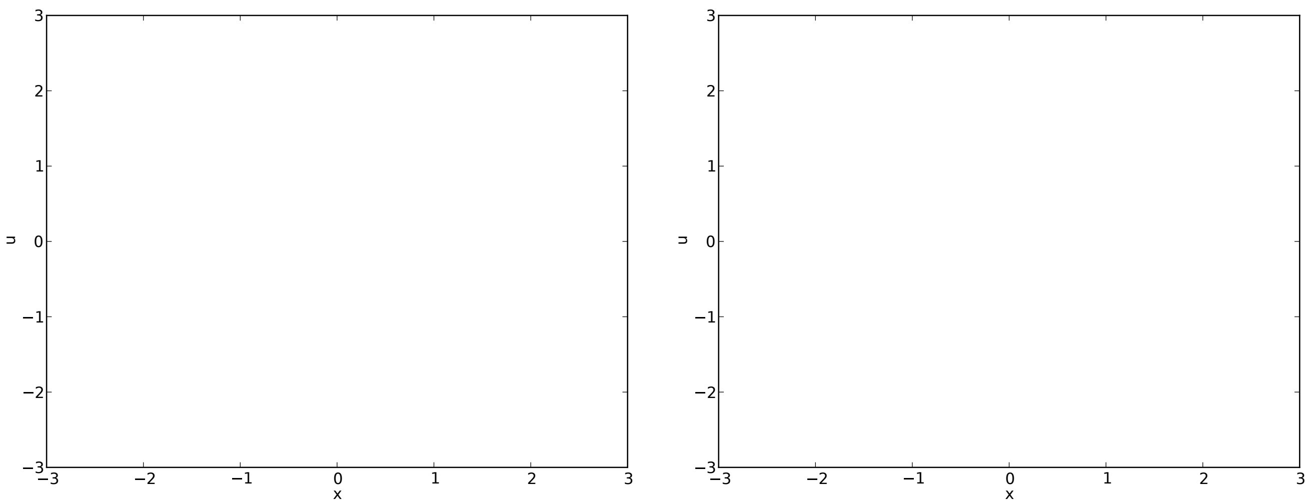

Figure 3: Direction field for the system of ODEs with $\gamma = 1/4$ and $\lambda = 1$. Two empty figures are provided in case you are not satisfied with your first attempt and want to produce a second. In case you do, use the second empty figure, make sure you cross out your worse attempt.

Last name, first name, student ID:

(e) Formulate the explicit Euler numerical scheme for the system of ODEs with the fixed $\gamma = 7/4$ and $\lambda = 2$. Under which conditions does the numerical scheme converge to the critical point?

2 Virus Mutation ($\approx$ 3 + 5 + 1 = 9 points)

Suppose a virus can exist in N different strains (species) and in each new generation it either stays the same or mutates to another strain. Among all N strains there is one especially dangerous strain. Therefore, we are only interested if the virus is in the dangerous strain or not. The probability that the dangerous strain mutates to another strain is $0 < \alpha < 1$ and that another strain mutates to exactly the dangerous strain is $0 < \beta < 1$.
The process is described by a Markov chain, whose diagram is presented in Figure 4.

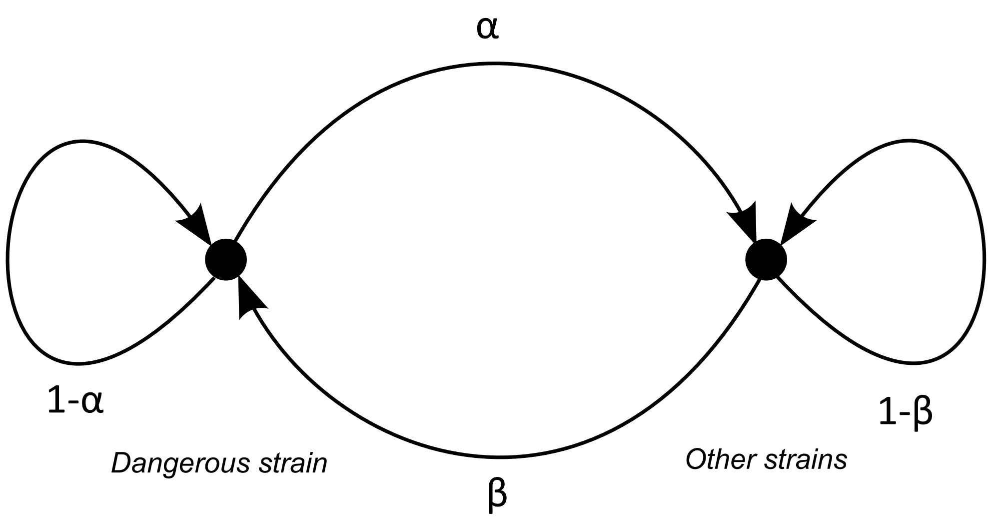

Figure 4: Two state virus mutation diagram.

The transition matrix describing the evolution of the probabilities is given by

$$P = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}. \tag{3}$$

The transition matrix allows us to find the probabilities $\mathbf{f}_{n+1}$ at time step $n+1$ to have the virus in the dangerous and other strains if we know the probabilities at time step n:

$$\mathbf{f}_{n+1} = P\mathbf{f}_n = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix},$$

where a_n, b_n are the probabilities of the dangerous and other strains at time step n.

(a) What is the probability distribution $\mathbf{f}_n$ in the long term limit $n \to \infty$?

Why does it not depend on the initial distribution?

Last name, first name, student ID:

(b) What is the probability that the virus is in the dangerous strain in the nth generation if initially (0th generation) the virus was also in the dangerous strain?

(c) Suppose that every virus strain mutates with the same probability α. Any of the left $N-1$ strains has the same probability to be the result of the mutation. What is the value of β in the transition matrix (3)?

Last name, first name, student ID:

3 FEM for the Diffusion Equation (≈ 3 + 3 + 7 + 1 = 14 points)

The time-dependent diffusion equation with homogeneous Dirichlet boundary conditions reads

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= D\Delta u(t,x) \quad \text{for } x \in \Omega, && (4)\\
u(t,x) &= 0 \quad \text{for } x \in \partial\Omega, && (5)\\
u(t=0,x) &= g(x), && (6)
\end{aligned}
$$

where $g(x)$ represents some prescribed initial value function and $D \in \mathbb{R}$ is a non-zero constant.

(a) Derive the weak form for the system (4)–(6) for test functions v which belong to some suitable function space V. No discretization of V is required. Use the integration by parts / divergence theorem to transform the 2nd-order derivative. Give a brief explanation why this transformation is useful!

(b) Discretization: Nodal assembling.
In the following, we are going to discretize the weak form of problem (4)–(6) in 2D using identical **linear** ansatz and test functions on triangular grid cells.
In Figure 5, a sketch of a specific triangular grid is given on a part of the domain Ω.

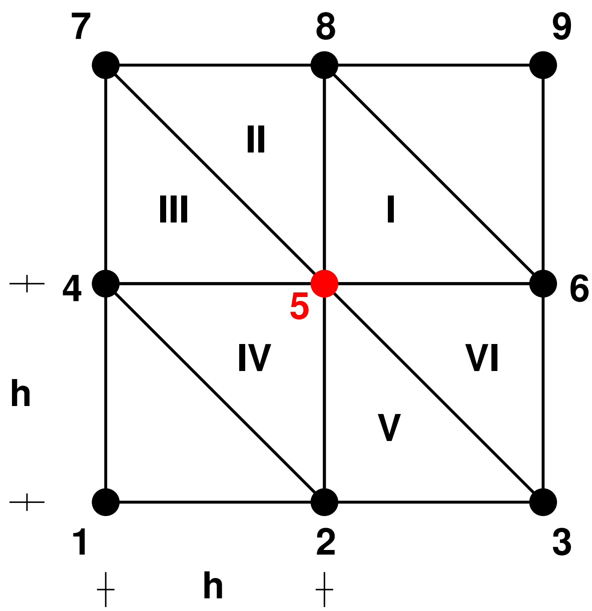

Figure 5: Special triangular mesh for linear ansatz and test functions on a part of the domain Ω. The dots and indices $1, \ldots, 9$ indicate the degrees of freedom, whereas the roman numbers I, . . . ,VI denote the triangles of mesh size h that contribute to the support of node 5.

Task: Use a linear nodal basis φ_i (i.e. $\varphi_i(x_j, y_j) = \delta_{ij}$) to identify (not necessarily compute) the derivative terms to fill Table 1.

	I	II	III	IV	V	VI
$\frac{\partial \varphi_5}{\partial x}$						
$\frac{\partial \varphi_5}{\partial y}$						

Table 1: Partial derivatives of ansatz function $\varphi_5(x, y)$ on the different triangles I, . . . ,VI of its support.

Last name, first name, student ID:

(c) Let A denote the stiffness matrix representing the discrete diffusion contribution of the above problem in its weak form. Compute the column entries $A_{5,j}, j = 1, \ldots, 9$ of row 5, i.e. the row for the center degree of freedom in Figure 5 involving its support I, . . . ,VI.
Hint: You may use symmetry arguments to shorten the calculations.

(d) How does the stencil of the diffusive term for node 5 look (with respect to all nine nodes $(1, \ldots, 9)$ in the sketched part of this specially structured grid of Figure 5) using the above FEM discretization? Interpret your result!

4 Stability Analysis of the Advection Equation (≈ 4 + 5 = 9 points)

Consider a one-dimensional linear advection equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \tag{7}$$

where v is the advection coefficient.

(a) Write down the discretized form of the advection equation. Use the explicit Euler scheme for the time discretization and the central difference approximation for the spatial derivative. Apply the von Neumann stability analysis and show that the numerical scheme is unconditionally unstable.

Last name, first name, student ID:

(b) Due to the instability of the numerical scheme in (a), one might try the forward or backward difference for the spatial derivative. Find the time step restriction from the von Neumann stability analysis for the forward difference. What is required for the advection coefficient to have the forward difference stable?