Repetition Exam Scientific Computing 1 (T. Neckel, D. Jarema) WS 2014/15	Page 1/14
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General Instructions

Material:

You may only use one hand-written sheet of paper (size A4, on both pages).

Any other material including electronic devices of any kind is forbidden.

Use only the exam paper that was handed out to solve the exercises. In case the space on a page is not enough, mark that you continue with your solution and use the reverse side of the preceding page. For additional notes and sketches, you can obtain additional exam sheets.

Do not use pencil, or red or green ink.

General hint:

Often, exercises b), c), etc. can be solved without the results from the previous exercise a); if you are stuck with exercise a), then don't immediately skip exercises b), c), etc.

Working time:

90 minutes + 5 minutes reading time.

Please switch off your cell phones!

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/11	/9	/13	/11	/≈ 44

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1 Streeter-Phelps Model for River Purification (≈ 3 + 4 + 4 = 11 points)

Streeter and Phelps proposed a simple model for decrease of the concentration of oxygen (due to polluting discharge) and its recovery to a background level in a river. The model is given by a system of ODEs

$$\begin{aligned} \frac{\partial c_p}{\partial t} &= f_p - k_1 c_p \\ \frac{\partial c_x}{\partial t} &= k_2 \left(c_{xs} - c_x \right) - k_1 c_p, \end{aligned} \tag{1}$$

where the following notation is used:

c_p	organic pollutant concentration
f_p	organic pollutant inflow rate
k_1	organic pollutant degradation rate
c_x	oxygen dissolved in water concentration
c_{xs}	oxygen dissolved in water saturation concentration
k_2	water reaeration rate

(a) What is the critical point of the model equations? What are the possible types of the critical point if parameters $k_1 > 0$ and $k_2 > 0$.

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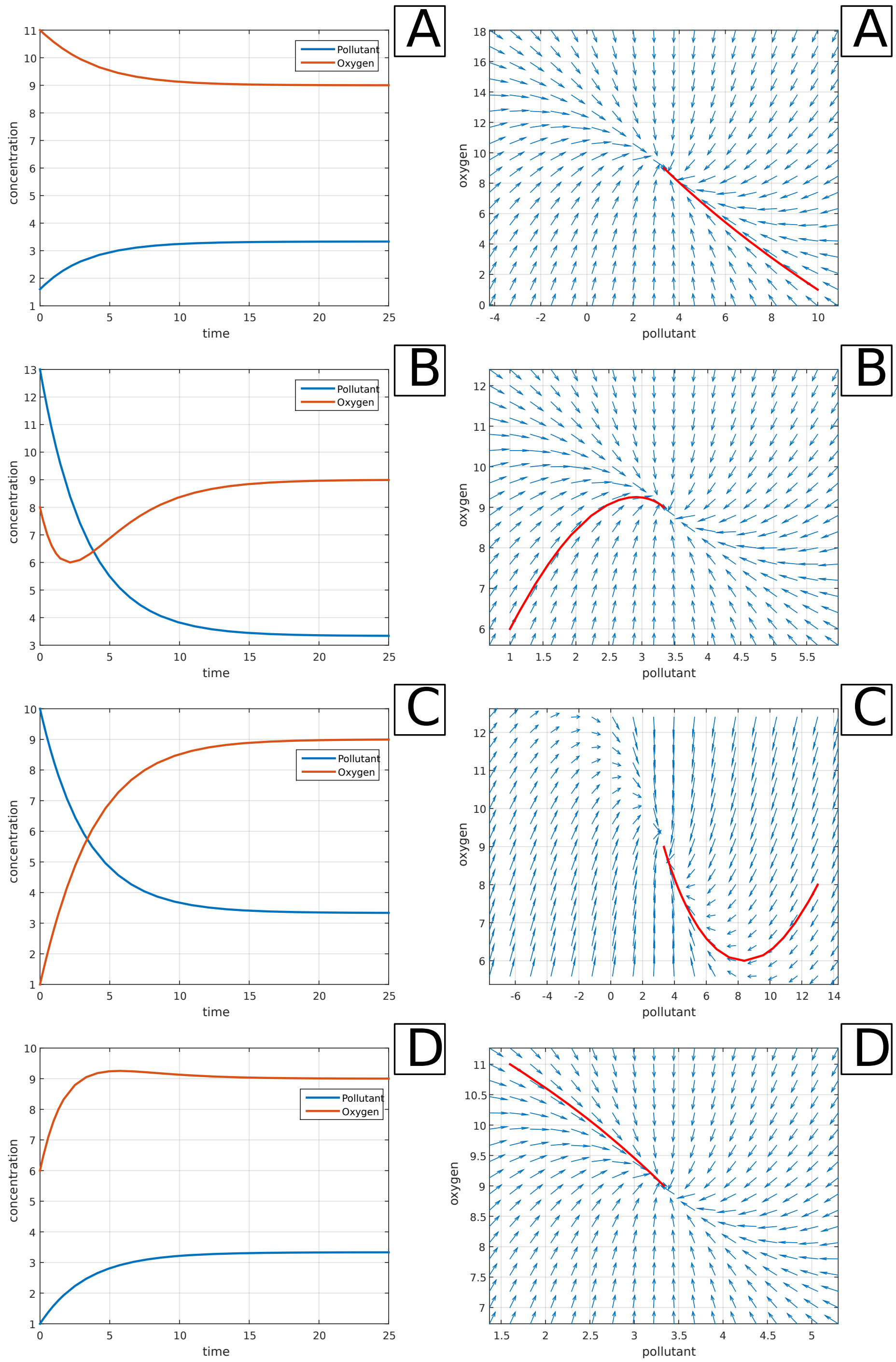

Figure 1: Time dependencies and phase trajectories of pollutant and oxygen concentrations for four different scenarios.

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(b) Time evolution of pollutant and oxygen concentrations for four different scenarios are shown in Figure 1 (left). For each of this time evolution find a plot in Figure 1 (right) with the corresponding trajectory in the phase-space.

(c) Formulate the implicit Euler method for the system of ODEs (1) using a time step τ. Compute the explicit form of the resulting update scheme for $c_p(t+\tau)$ and $c_x(t+\tau)$.

Hint: you may use the following formula to invert 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

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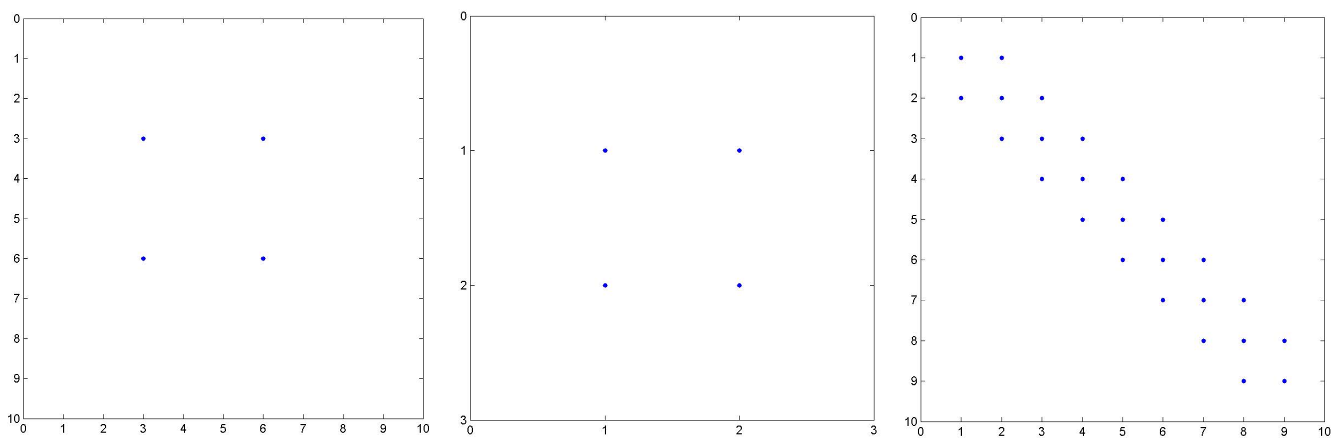
2 FEM - 1D Poisson problem (≈ 1.5+0.5+1+1.5+0.5+4 = 9 points)

In this task, we consider the 1D Poisson problem with Dirichlet-zero boundary conditions. We apply a FEM discretization with linear nodal base functions.

(a) For $N = 10$ cells, the following three images visualize the structure of three matrices (i.e. each dot indicates a non-zero element):



(a) pic 1 (b) pic 2 (c) pic 3

Figure 2: Visualization of the structure of 3 different matrices relevant for the FEM construction of the problem.

Map the 3 images of Fig. to the three different matrices:

I) global stiffness matrix A

II) global element stiffness matrix A^k

III) local element stiffness matrix $\hat{A}^k$

(b) What is the dimension of the global system to be solved if N cells are used?

(c) Why is the weak form called weak?

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(d) Let $\phi(x)$ be the standard linear hat functions on an equidistant grid with mesh size $h = 1/N$ in the domain $\Omega = (0, 1)$. The kth element (cell) in the mesh is given by

$$\Omega^{(k)} := h \cdot [k, k+1] \; .$$

For $x \in [0, 1]$, indicate the formulas of the basis functions $\phi_1^{(k)}(x)$ and $\phi_2^{(k)}(x)$ on element $\Omega^{(k)}$.

(e) What is the dimension of the element stiffness matrix $\widehat{A}_{\mu,\nu}^{(k)}$?

(f) Compute the element stiffness matrix $\widehat{A}_{\mu,\nu}^{(k)}$ for our special setup of an equidistant grid.

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3 Professor and Driving License (≈ 2 + 4 + 5 + 2 = 13 points)

Professor Schmidt is traveling to work by car. Sometimes he forgets to take his driving license. The probability that the professor forgets his driving license is p. We distinguish three different situations:

1. The professor has his driving license with him in the car.
2. The professor has his driving license at the point of departure.
3. The professor has his driving license at his destination.

The diagram describing these three situations and showing the conditional probabilities is demonstrated in Figure 3.

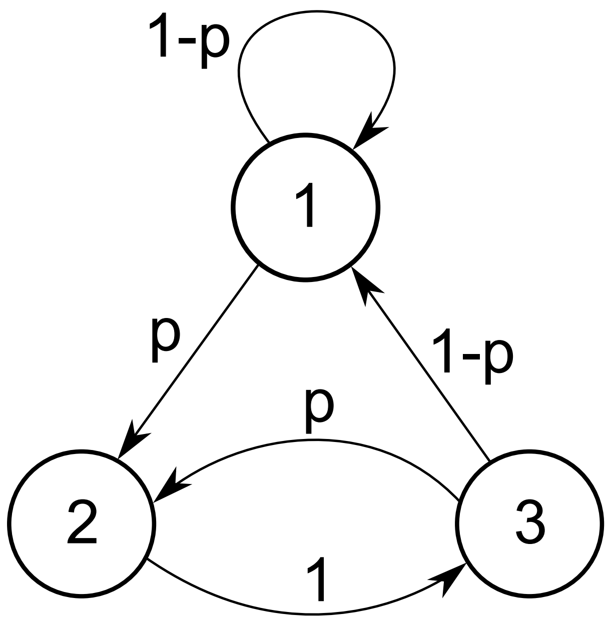

Figure 3: Scheme of three different situations for the professor's way to home or to office.

The transition matrix describing the evolution of probabilities of each of the three situations is given by

$$\hat{P} = \begin{pmatrix} 1-p & 0 & 1-p \\ p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

The transition matrix allows us to find the probabilities $\mathbf{x}_{n+1}$ at time step $n+1$ if we know the probabilities at time step n:

$$\mathbf{x}_{n+1} = \hat{P}\mathbf{x}_n \tag{4}$$

(a) We know that the professor returned home with the driving license. What are the probabilities of all three situations on his return home the next day?

(b) Show that it does not matter what the initial probabilities $\mathbf{x}_0$ are, in the long term limit $n \to \infty$ the probabilities vector $\mathbf{x}_n$ will always converge to the stationary solution. Find this stationary solution.

(c) After the Easter vacation, the professor's wife took measures and made sure that the professor has the driving license with him on his first trip to work. Furthermore, we know that the probability that the professor forgets his driving license is $p = 0.5$ and the transition matrix has the form

$$\hat{P} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

The expected number of times when the professor has his driving license with him after n trips is computed by formula

$$N = \sum_{m=0}^{n-1} \mathbf{c}^\intercal \mathbf{x}_m, \tag{6}$$

where vector $\mathbf{c} = (1, 0, 0)^\intercal$.

Show that $N(n) = \frac{n}{3} + \frac{7}{9} + \frac{2}{9}\left(-\frac{1}{2}\right)^n$.

Hint: You may use the eigenvalue decomposition of the transaction matrix:

$$\hat{P} = \hat{V}\hat{D}\hat{V}^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{7}$$

And the geometric series:

$$\sum_{m=0}^{n-1} ar^m = a\frac{1 - r^n}{1 - r}. \tag{8}$$

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(d) Now and then, the professor is checked with probability α by the road police. If during the check the professor finds out that his driving license is at the point of departure, he returns and the next time drives with the license, otherwise he drives further and does not forget to take the driving license for the next trip.

Extend the diagram of the initial model and write the corresponding probabilities in Figure 4.

Construct the corresponding transition matrix.

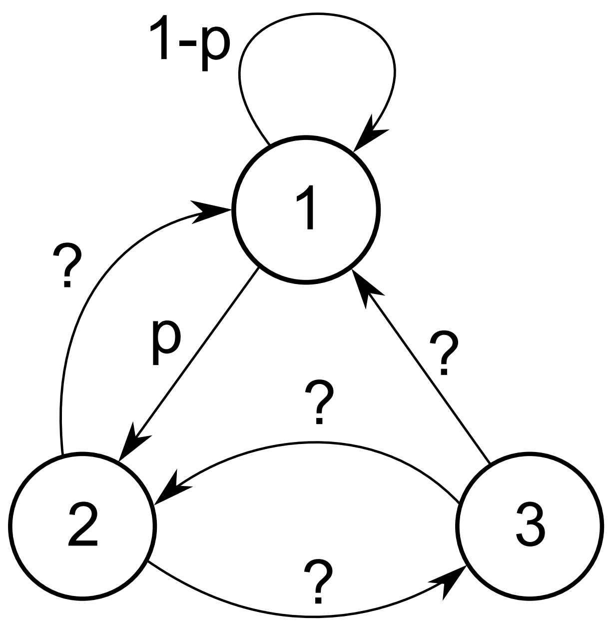

Figure 4: Scheme of three different situations on professor's way to home or office with possible police check.

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4 Artificial Viscosity for the Advection Equation (≈ 5 + 6 = 11 points)

Artificial viscosity is sometimes used to achieve a well-behaved numerical solution close to discontinuities. As a consequence, the artificial viscosity influences the stability of the modified numerical scheme. Consider a one-dimensional linear advection equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \tag{9}$$

where v is the advection coefficient.

(a) Write down the discretized form of the advection equation. Use the explicit Euler scheme for the time discretization and the central difference approximation for the spatial derivative. Add an artificial viscosity to the right hand side of the equation. For the artificial viscosity use the following expression

$$\epsilon\frac{\partial^2 u}{\partial x^2},$$

where ϵ is an artificial viscosity coefficient. For the artificial viscosity discretization also use the central difference scheme.

Apply the von Neumann stability analysis and show that the error magnitude takes the form:

$$|a_k|^2 = (1 - \lambda\epsilon + \lambda\epsilon\cos\phi)^2 + (\lambda v \sin\phi)^2, \tag{10}$$

where λ is the time step size divided by the mesh size $\lambda = \tau/h$ and $\phi = \pi k h$.

Hint: Use the error of type

$$\epsilon_j^{(n)} = a_k^n e^{\mathrm{i}\pi hkj}.$$

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(b) What is a right choice for the artificial viscosity coefficient in task (a) that makes the numerical scheme stable?

Hint: Make sure that $|a_k|^2 \leq 1$ at $\phi = \pm\pi$. Furthermore, $|a_k|^2(\phi)$ function has an extremum point at $\phi = 0$ and $|a_k|^2(\phi = 0) = 1$, to make sure that this is a maximum point impose that the second order derivative:

$$\frac{\partial^2 |a_k|^2}{\partial \phi^2} \leq 0.$$

You may need the derivatives of the sine and cosine functions:

$$\frac{\mathrm{d} \sin x}{\mathrm{d}x} = \cos x,$$
$$\frac{\mathrm{d} \cos x}{\mathrm{d}x} = -\sin x.$$

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