

Outlines

Part I: Discrete Models

Part II: A Continuous Model
– The Heat Equation

Scientific Computing I

Module 5: Heat Transfer – Discrete and Continuous Models

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Winter 2005/2006

Outlines

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Part II: A Continuous Model
– The Heat Equation

Part I: Discrete Models

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Part II: A Continuous Model – The Heat Equation

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Part I

Discrete Models

Motivation: Heat Transfer

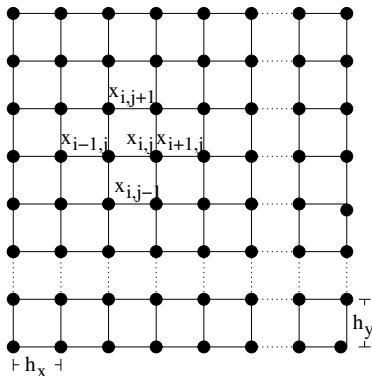
- objective: compute the temperature distribution of some object
- under certain prerequisites:
 - temperature at object boundaries given
 - heat sources
 - material parameters
- observation from physical experiments:

$$q \approx k \cdot \delta T$$

heat flow proportional to temperature differences

A Wiremesh Model

- consider rectangular plate as fine mesh of wires
- compute temperature x_{ij} at nodes of the mesh



Wiremesh Model (2)

- model assumption: temperatures in equilibrium at every mesh node
- for all temperatures x_{ij} :

$$x_{ij} = \frac{1}{4} (x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1})$$

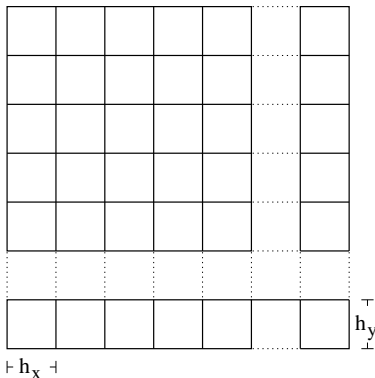
- temperature known at (part of) the boundary;
for example:

$$x_{0,j} = T_j$$

- task: solve system of linear equations

A Finite Volume Model

- object: e.g. a rectangular metal plate
- model as a collection of small connected rectangular cells



- examine the heat flow across the cell edges

Heat Flow Across the Cell Boundaries

- Heat flow across a given edge is proportional to
 - temperature difference ($T_1 - T_0$) between the adjacent cells
 - length h of the edge
- e.g.: heat flow across the left edge:

$$q_{ij}^{(\text{left})} = k_x (T_{ij} - T_{i-1,j}) h_y$$

- heat flow across all edges determines change of heat energy:

$$\begin{aligned} q_{ij} &= k_x (T_{ij} - T_{i-1,j}) h_y + k_x (T_{ij} - T_{i+1,j}) h_y \\ &+ k_y (T_{ij} - T_{i,j-1}) h_x + k_y (T_{ij} - T_{i,j+1}) h_x \end{aligned}$$

Temperature change due to heat flow

- in equilibrium: total heat flow equal to 0
- but: consider additional source term F_{ij} due to
 - external heating
 - radiation
- $F_{ij} = f_{ij}h_xh_y$ (f_{ij} heat flow per area)
- equilibrium with source term requires $q_{ij} + F_{ij} = 0$:

$$\begin{aligned} f_{ij}h_xh_y &= -k_xh_y(2T_{ij} - T_{i-1,j} - T_{i+1,j}) \\ &\quad -k_yh_x(2T_{ij} - T_{i,j-1} - T_{i,j+1}) \end{aligned}$$

Finite Volume Model

- divide by $h_x h_y$:

$$f_{ij} = -\frac{k_x}{h_x} (2T_{ij} - T_{i-1,j} - T_{i+1,j}) - \frac{k_y}{h_y} (2T_{ij} - T_{i,j-1} - T_{i,j+1})$$

- again, system of linear equations
- how to treat boundaries?
 - prescribe temperature in a cell (e.g. boundary layer of cells)
 - prescribe heat flow across an edge; for example insulation at left edge:

$$q_{ij}^{(\text{left})} = 0$$

A Time Dependent Model

- idea: set up ODE for each cell
- no external heat sources or drains: $f_{ij} = 0$
- change of temperature per time is proportional to heat flow into the cell (no longer 0):

$$\begin{aligned}\dot{T}_{ij}(t) &= \frac{\kappa_x}{h_x} (2T_{ij}(t) - T_{i-1,j}(t) - T_{i+1,j}(t)) \\ &+ \frac{\kappa_y}{h_y} (2T_{ij}(t) - T_{i,j-1}(t) - T_{i,j+1}(t))\end{aligned}$$

- solve system of ODE

Part II

A Continuous Model – The Heat Equation

From Discrete to Continuous

- remember the discrete model:

$$f_{ij} = -\frac{k_x}{h_x} (2T_{ij} - T_{i-1,j} - T_{i+1,j}) - \frac{k_y}{h_y} (2T_{ij} - T_{i,j-1} - T_{i,j+1})$$

- assumption: heat flow across edges is proportional to temperature *difference*

$$q_{ij}^{(\text{left})} = k_x (T_{ij} - T_{i-1,j}) h_y$$

- in reality: heat flow proportional to temperature *gradient*

$$q_{ij}^{(\text{left})} \approx kh_y \frac{T_{ij} - T_{i-1,j}}{h_x}$$

From Discrete to Continuous (2)

- replace k_x by k/h_x , k_y by k/h_y , and get:

$$f_{ij} = -\frac{k}{h_x^2} (2T_{ij} - T_{i-1,j} - T_{i+1,j}) \\ -\frac{k}{h_y^2} (2T_{ij} - T_{i,j-1} - T_{i,j+1})$$

- consider arbitrary small cells: $h_x, h_y \rightarrow 0$:

$$f_{ij} = -k \left(\frac{\partial^2 T}{\partial x^2} \right)_{ij} - k \left(\frac{\partial^2 T}{\partial y^2} \right)_{ij}$$

- leads to *partial differential equation* (PDE):

$$-k \left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} \right) = f(x,y)$$

Derivation of the Heat Equation

- finite volume model, but with arbitrary control volume D
- change of heat energy (per time) is a result of
 - transfer of heat energy across D 's surface,
 - heat sources and drains in D (external influences)
- resulting integral equation:

$$\frac{\partial}{\partial t} \int_D \rho c T dV = \int_D q dV + \int_{\partial D} k \nabla T \cdot \vec{n} dS$$

density ρ , specific heat c , and heat conductivity k are material parameters

- heat sources and drains are modelled in term q

Derivation of the Heat Equation (2)

- according to theorem of Gauß:

$$\int_{\partial D} k \nabla T \cdot \vec{n} \, dS = \int_D k \Delta T \, dV$$

- leads to integral equation for **any** domain D :

$$\int_D \rho c T_t - q - k \Delta T \, dV = 0$$

- hence, the integrand has to be identically 0:

$$T_t = \kappa \Delta T + \frac{q}{\rho c}, \quad \kappa := \frac{k}{\rho c}$$

- $\kappa > 0$ is called the *thermal diffusion coefficient* (since the Laplace operator models a (heat) diffusion process)

Heat Equations

Different scenarios:

- vanishing external influence, $q = 0$:

$$T_t = \kappa \Delta T$$

alternate notation

$$\frac{\partial T}{\partial t} = \kappa \cdot \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

- equilibrium solution, $T_t = 0$:

$$0 = \kappa \Delta T + \frac{q}{\rho c} \quad \longrightarrow \quad -\Delta T = f$$

“Poisson’s Equation”

Boundary Conditions

Dirichlet boundary conditions:

- fix T on (part of) the boundary

$$T(x,y,z) = \varphi(x,y,z)$$

Neumann boundary conditions:

- fix T 's normal derivative on (part of) the boundary:

$$\frac{\partial T}{\partial n}(x,y,z) = \varphi(x,y,z)$$

- special case: insulation

$$\frac{\partial T}{\partial n}(x,y,z) = 0$$