Scientific Computing I

Module 5: Heat Transfer – Discrete and Contiuous Models

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Part I

Discrete Models

Motivation: Heat Transfer

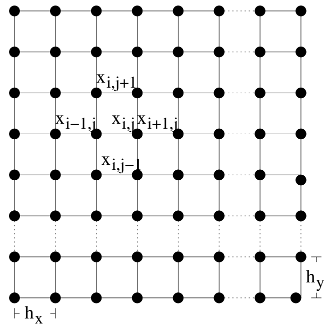
- objective: compute the temperature distribution of some object
- under certain prerequisites:
  - temperature at object boundaries given
  - heat sources
  - material parameters
- observation from physical experiments:

$$q \approx k \cdot \delta T$$

heat flow proportional to temperature differences

A Wiremesh Model

- consider rectangular plate as fine mesh of wires
- compute temperature x_{ij} at nodes of the mesh



Wiremesh Model (2)

- model assumption: temperatures in equilibrium at every mesh node
- for all temperatures x_{ij}:

$$x_{ij} = \frac{1}{4}\left(x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1}\right)$$

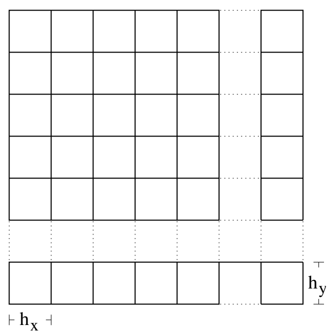
- temperature known at (part of) the boundary; for example:

$$x_{0,j} = T_j$$

- task: solve system of linear equations

A Finite Volume Model

- object: e.g. a rectangular metal plate
- model as a collection of small connected rectangular cells



- examine the heat flow across the cell edges

Heat Flow Across the Cell Boundaries

- Heat flow across a given edge is proportional to
  - temperature difference $(T_1 - T_0)$ between the adjacent cells
  - length h of the edge
- e.g.: heat flow across the left edge:

$$q_{ij}^{(\text{left})} = k_x \left(T_{ij} - T_{i-1,j}\right) h_y$$

- heat flow across all edges determines change of heat energy:

$$\begin{aligned} q_{ij} &= k_x \left(T_{ij} - T_{i-1,j}\right) h_y + k_x \left(T_{ij} - T_{i+1,j}\right) h_y \\ &+ k_y \left(T_{ij} - T_{i,j-1}\right) h_x + k_y \left(T_{ij} - T_{i,j+1}\right) h_x \end{aligned}$$

Temperature change due to heat flow

- in equilibrium: total heat flow equal to 0
- but: consider additional source term F_{ij} due to
  - external heating
  - radiation
- $F_{ij} = f_{ij} h_x h_y$ (f_{ij} heat flow per area)
- equilibrium with source term requires $q_{ij} + F_{ij} = 0$:

$$\begin{aligned} f_{ij} h_x h_y &= -k_x h_y \left(2T_{ij} - T_{i-1,j} - T_{i+1,j}\right) \\ &\quad -k_y h_x \left(2T_{ij} - T_{i,j-1} - T_{i,j+1}\right) \end{aligned}$$

Finite Volume Model

- divide by $h_x h_y$:

$$\begin{aligned} f_{ij} &= -\frac{k_x}{h_x} \left(2T_{ij} - T_{i-1,j} - T_{i+1,j}\right) \\ &\quad -\frac{k_y}{h_y} \left(2T_{ij} - T_{i,j-1} - T_{i,j+1}\right) \end{aligned}$$

- again, system of linear equations
- how to treat boundaries?
  - prescribe temperature in a cell (e.g. boundary layer of cells)
  - prescribe heat flow across an edge; for example insulation at left edge:

$$q_{ij}^{(\text{left})} = 0$$

Part II

A Continuous Model – The Heat Equation

A Time Dependent Model

- idea: set up ODE for each cell
- no external heat sources or drains: $f_{ij} = 0$
- change of temperature per time is proportional to heat flow into the cell (no longer 0):

$$\begin{aligned} \dot{T}_{ij}(t) &= \frac{\kappa_x}{h_x} \left(2T_{ij}(t) - T_{i-1,j}(t) - T_{i+1,j}(t)\right) \\ &+ \frac{\kappa_y}{h_y} \left(2T_{ij}(t) - T_{i,j-1}(t) - T_{i,j+1}(t)\right) \end{aligned}$$

- solve system of ODE

From Discrete to Contiuous

- remember the discrete model:

$$\begin{aligned} f_{ij} &= -\frac{k_x}{h_x} \left(2T_{ij} - T_{i-1,j} - T_{i+1,j}\right) \\ &\quad -\frac{k_y}{h_y} \left(2T_{ij} - T_{i,j-1} - T_{i,j+1}\right) \end{aligned}$$

- assumption:heat flow accross edges is proportional to temperature *difference*

$$q_{ij}^{(\text{left})} = k_x \left(T_{ij} - T_{i-1,j}\right) h_y$$

- in reality: heat flow proportional to temperature *gradient*

$$q_{ij}^{(\text{left})} \approx k h_y \frac{T_{ij} - T_{i-1,j}}{h_x}$$

From Discrete to Contiuous (2)

- replace k_x by k/h_x, k_y by k/h_y, and get:

$$\begin{aligned} f_{ij} &= -\frac{k}{h_x^2}\left(2T_{ij} - T_{i-1,j} - T_{i+1,j}\right) \\ &\quad -\frac{k}{h_y^2}\left(2T_{ij} - T_{i,j-1} - T_{i,j+1}\right) \end{aligned}$$

- consider arbitrary small cells: $h_x, h_y \to 0$:

$$f_{ij} = -k\left(\frac{\partial^2 T}{\partial x^2}\right)_{ij} - k\left(\frac{\partial^2 T}{\partial y^2}\right)_{ij}$$

- leads to *partial differential equation* (PDE):

$$-k\left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2}\right) = f(x,y)$$

Derivation of the Heat Equation (2)

- according to theorem of Gauß:

$$\int_{\partial D} k\nabla T \cdot \vec{n}\,\mathrm{d}S = \int_{D} k\Delta T\,\mathrm{d}V$$

- leads to integral equation for **any** domain D:

$$\int_{D} \rho c T_t - q - k\Delta T\,\mathrm{d}V = 0$$

- hence, the integrand has to be identically 0:

$$T_t = \kappa\Delta T + \frac{q}{\rho c}, \qquad \kappa := \frac{k}{\rho c}$$

- $\kappa > 0$ is called the *thermal diffusion coefficient* (since the Laplace operator models a (heat) diffusion process)

Boundary Conditions

Dirichlet boundary conditions:

- fix T on (part of) the boundary

$$T(x,y,z) = \varphi(x,y,z)$$

Neumann boundary conditions:

- fix T's normal derivative on (part of) the boundary:

$$\frac{\partial T}{\partial n}(x,y,z) = \varphi(x,y,z)$$

- special case: insulation

$$\frac{\partial T}{\partial n}(x,y,z) = 0$$

Derivation of the Heat Equation

- finite volume model, but with arbitrary control volume D
- change of heat energy (per time) is a result of
  - transfer of heat energy across D's surface,
  - heat sources and drains in D (external influences)
- resulting integral equation:

$$\frac{\partial}{\partial t}\int_{D} \rho c T\,\mathrm{d}V = \int_{D} q\,\mathrm{d}V + \int_{\partial D} k\nabla T \cdot \vec{n}\,\mathrm{d}S$$

  density ρ, specific heat c, and heat conductivity k are material parameters
- heat sources and drains are modelled in term q

Heat Equations

Different scenarios:

- vanishing external influence, $q = 0$:

$$T_t = \kappa\Delta T$$

  alternate notation

$$\frac{\partial T}{\partial t} = \kappa \cdot \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

- equilibrium solution, $T_t = 0$:

$$0 = \kappa\Delta T + \frac{q}{\rho c} \quad \longrightarrow \quad -\Delta T = f$$

  "Poisson's Equation"