Scientific Computing I

Module 3: Population Modelling – Continuous Models

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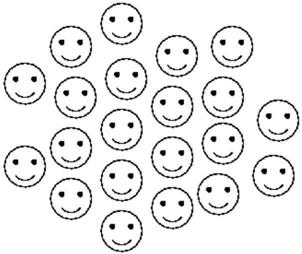
Part I: ODE Models

Part II: More Than One Species – Systems of ODE

Part I

ODE Models

Discrete vs. Contiuous Models



$$\frac{dp}{dt} = F(p, t, \dots)$$

$$p(t) = ?$$

discrete model:
$p(t) \in \mathbb{N}$ individuals

continuous model:
$p : \mathbb{R} \rightarrow \mathbb{R}, p(t) = ?$

Advantage:

- easier(?) type of mathematical problem: differential equations, calculus
- analytical solutions available(?)

Model of Maltus (1798)

Only one species:

1. birth rate γ (number of births per time interval) proportional to size of population
2. death rate δ proportional to size of population
3. thus: constant growth (or decay) rate: $r = \gamma - \delta$

Modelling:

- constant growth rate

$$\frac{dp}{dt} = r \cdot p$$

- growth within a time interval

$$p(t + \delta t) = p(t) + r \cdot p(t) \cdot \delta t$$

Model of Maltus – Differential Equation

Written as an ordinary differential equation:

$$\dot{p}(t) = r \cdot p(t)$$

Requires initial condition (population at start)

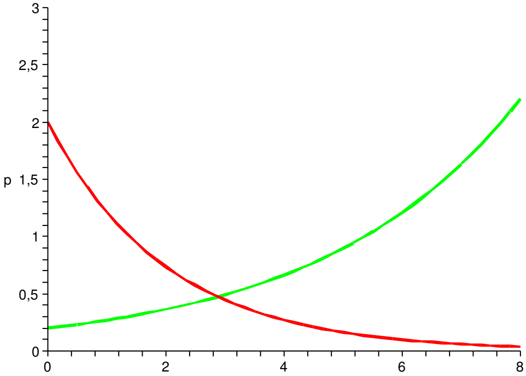
$$p(0) = p_0$$

Analytical solution:

$$p(t) = p_0 e^{rt}$$

Model of Maltus – Solutions

The model of maltus describes exponential growth or decay of a population:



Model of Verhulst (19th century)

Objective:

- model populations that approach saturation value

Assumptions:

- growth/death rate depend on population size; assume linear dependency:

$$g(t) = g_0 - g_1 \cdot p(t) \qquad d(t) = d_0 + d_1 \cdot p(t)$$

- leads to differential equation:

$$\dot{p}(t) = g(t) - d(t) = \underbrace{(g_0 - d_0)}_{=:\alpha} - \underbrace{(g_1 + d_1)}_{=:\beta} p(t)$$

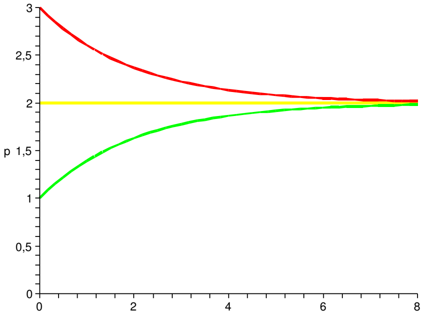
Model of Verhulst – Saturation

- solve initial value problem:

$$\dot{p}(t) = \alpha - \beta p(t), \qquad p(0) = p_0$$

- solution:

$$p(t) = p_\infty + e^{-\beta t}\left(p_0 - p_\infty\right), \qquad p_\infty = \frac{\alpha}{\beta}$$



Model of Verhulst – Logistic Growth

- saturation model does no longer model exponential growth
- let growth/death rate decrease linearly with time
- but keep growth/death rate proportinal to population size
- leads to differential equation:

$$\dot{p}(t) = (\alpha - \beta p(t))\, p(t)$$

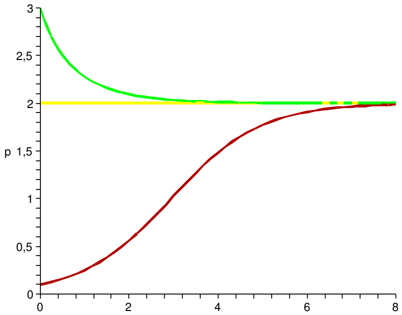
Logistic Growth

- other formulation

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

- solution:

$$p(t) = \frac{\beta}{(1 - e^{-\alpha t}) + \frac{\beta}{p_0} e^{-\alpha t}}$$

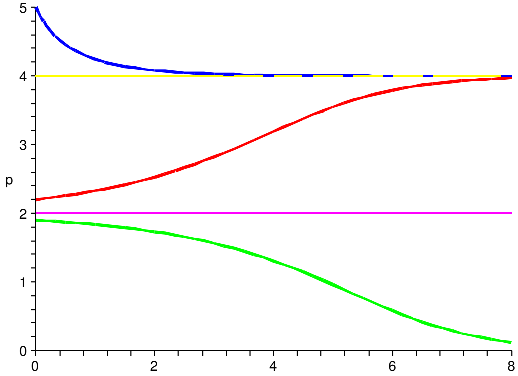


Logistic Growth with Threshold

- extended version of Verhulst's model:

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) \left(1 - \frac{p(t)}{\delta}\right) p(t)$$

- solutions ($\beta = 2, \delta = 4$):



Example – The Passenger Pigeon

- beginning of the 19th century, estimated population in North America: four billion
- hunting diminished its number below a critical threshold (late 1880s)
- The last passenger pigeon died on September, 1st 1914.

Part II

More Than One Species – Systems of ODE

A Linear Model

- similar to Verhulst's saturation model
- additional growth term proportional to other species
- leads to system of differential equations:

$$\dot{p}(t) = b_1 + a_{11}p(t) + a_{12}q(t)$$
$$\dot{q}(t) = b_2 + a_{21}p(t) + a_{22}q(t)$$

- typically:
  - $b_1 > 0, b_2 > 0$ (growth term)
  - $a_{11} < 0, a_{22} < 0$ (saturation)
  - a_{12}, a_{21}?

First Example: Arms Race

- armament of two (hostile) countries
- our suspicion: $a_{12} > 0$, $a_{21} > 0$

Observation:

- long-time behaviour depends on size of parameters
- steady-state solutions exist
- solutions exist that show unlimited growth

Second Example: Competition

- two species sharing a common natural habitat
- competition: $a_{12} < 0$, $a_{21} < 0$

Observation:

- long-time behaviour depends on size of parameters
- steady-state solutions exist
- some scenarios are physically incorrect! (negative population size)

A Non-Linear Model

- similar to Verhulst's logistic growth model
- additional growth term proportional to other species
- leads to system of differential equations:

$$\begin{aligned}\dot{p}(t) &= (b_1 + a_{11}p(t) + a_{12}q(t))\,p(t)\\ \dot{q}(t) &= (b_2 + a_{21}p(t) + a_{22}q(t))\,q(t)\end{aligned}$$

- typically:
  - $b_1 > 0, b_2 > 0$ (growth term)
  - $a_{11} < 0, a_{22} < 0$ (saturation)
  - a_{12}, a_{21}?

The Non-Linear Competition Model

- two species sharing a common natural habitat
- competition: $a_{12} < 0$, $a_{21} < 0$

Possible Scenarios:

- steady-state
- one species dies out (extinction)
- no obvious nonsense

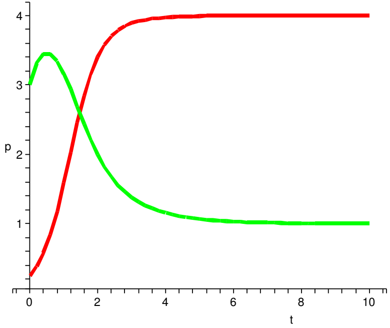
Competition – Steady State

- system of differential equations:

$$\dot{p}(t) = \left(\frac{5}{2} + \frac{\sqrt{3}}{24} - \frac{5}{8}p(t) - \frac{\sqrt{3}}{24}q(t)\right)p(t)$$

$$\dot{q}(t) = \left(\frac{7}{8} + \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{8}p(t) - \frac{7}{8}q(t)\right)q(t)$$

- solution for $p_0 = \frac{1}{4}$, $q_0 = 3$:

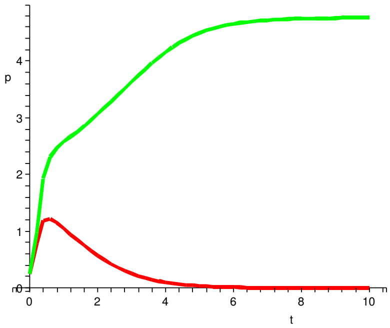


Competition – Extinction

- system of differential equations:

$$\dot{p}(t) = \left(\frac{71}{8} - \frac{23}{12}p(t) - \frac{25}{12}q(t)\right)p(t)$$
$$\dot{q}(t) = \left(\frac{73}{8} - \frac{25}{12}p(t) - \frac{23}{12}q(t)\right)q(t)$$

- solution for $p_0 = \frac{1}{4}$, $q_0 = \frac{1}{4}$:



Predator-Prey

- two species: predator p and prey q
- predator eats prey: $a_{12} > 0$
- prey is eaten by predator: $a_{21} < 0$

Possible Scenarios:

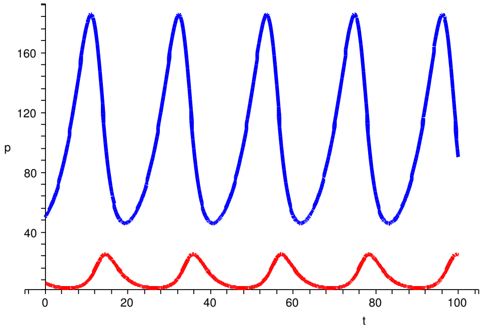
- stable oscillations
- one species dies out (what happens with the other, then?)

Predator-Prey by Lotka & Volterra

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= \left(-\tfrac{1}{2} + \tfrac{1}{200} q(t)\right) p(t) \\ \dot{q}(t) &= \left(\tfrac{1}{5} - \tfrac{1}{50} p(t)\right) q(t)\end{aligned}$$

- solution for $p_0 = 6$, $q_0 = 50$:



Open Questions

Methods to Analyse a Given Model?

- predict approximate solution or shape of solution?
- predict possible steady states?
- predict critical points?
  (species on edge of extiction?)

Methods to Improve Modeling?

- predict failure of the model?
- tune parameters to model a specific situation?