Scientific Computing I

Module 3: Population Modelling – Continuous Models (Part III)

Michael Bader

Lehrstuhl Informatik V

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Part III

Discussion and Analysis of ODE Models

Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
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Stability of Non-Linear Systems

Analysing the Slope of a Solution

Example: Model of Maltus

$$\dot{p}(t) = \alpha p(t)$$

- for a sensible solution: $p(t) > 0$
- α decides slope of solution:
  - $\alpha > 0$: growing population (accelerated growth)
  - $\alpha < 0$: receding population (decelerated reduction)

Critical Points
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Critical Points in 1D
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Analysis of Systems of ODE
Homogeneous Systems
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Points of Equilibrium

Example: Model of Verhulst (saturation)

$$\dot{p}(t) = \alpha - \beta p(t)$$

- equilibrium: $\dot{p}(t) = 0$
- only, if $p(t) = \frac{\alpha}{\beta}$

Example: Logistic Growth

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

- constant solution, if $p(t) = \beta$ or $p(t) = 0$

Critical Points

Observation on Logistic Growth:

- constant solution $p(t) = \beta$, if $p(0) = \beta$
- constant solution $p(t) = 0$, if $p(0) = 0$
- equilibrium at $p = \beta$ is reached for nearly all initial conditions
  $\Rightarrow$ *attractive* (stable) equilibrium
- equilibrium at $p = 0$ is not reached for any other initial conditions ("repulsive")
  $\Rightarrow$ unstable equilibrium

Critical Points – Derivatives

Examine derivatives:

- critical point $p = \bar{p}$
- attractive equilibrium (asymptotically stable):

$$\begin{aligned} \dot{p} < 0 \quad &\text{for} \quad p = \bar{p} + \varepsilon \\ \dot{p} > 0 \quad &\text{for} \quad p = \bar{p} - \varepsilon \end{aligned}$$

- unstable equilibrium:

$$\begin{aligned} \dot{p} > 0 \quad &\text{for} \quad p = \bar{p} + \varepsilon \\ \dot{p} < 0 \quad &\text{for} \quad p = \bar{p} - \varepsilon \end{aligned}$$

- otherwise: saddle point

Critical Points
Points of Equilibrium
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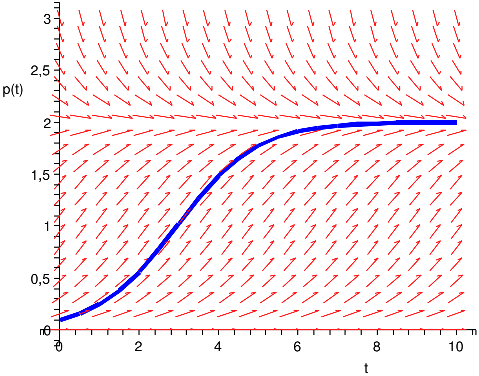
Direction Field

plot derivatives vs. time and size of population:

Example: Logistic Growth

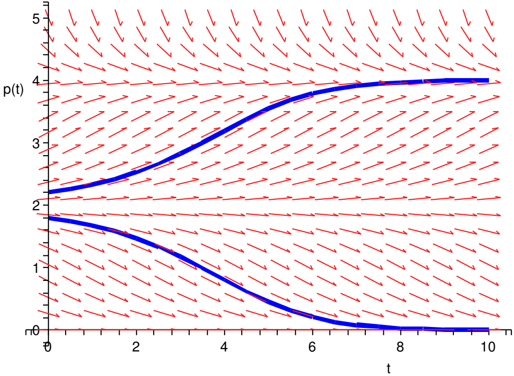
$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$



Direction Field (2)

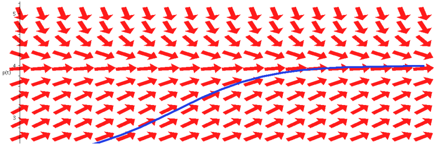
Example: Logistic Growth with Threshold

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right)\left(1 - \frac{p(t)}{\delta}\right) p(t)$$

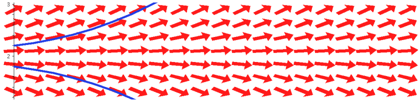


Identifying Critical Points

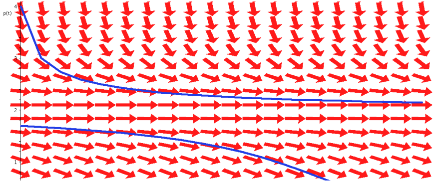
- attractive equilibrium:



- unstable equilibrium



- saddle point



Critical Points in 2D

Example: Arms Race

- system of differential equations
- equilibrium: $\dot{p} = 0$, $\dot{q} = 0$

$$
\begin{aligned}
\dot{p}(t) &= b_1 + a_{11} p(t) + a_{12} q(t) = 0 \\
\dot{q}(t) &= b_2 + a_{21} p(t) + a_{22} q(t) = 0
\end{aligned}
$$

- solution of a linear system of equations:

$$
\begin{aligned}
a_{11} p(t) + a_{12} q(t) &= -b_1 \\
a_{21} p(t) + a_{22} q(t) &= -b_2
\end{aligned}
$$

- in most cases one critical point
- critical line, if system matrix is singular

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Direction Field for a System of ODE

- example: 2D system of differential equations:

$$\begin{aligned} \dot{p}(t) &= b_1 + a_{11}p(t) + a_{12}q(t) \\ \dot{q}(t) &= b_2 + a_{21}p(t) + a_{22}q(t) \end{aligned}$$

- natural exension: 3D plot: t vs. p vs. q
- 1D direction field for p vs. t or q vs. t not sufficient:
  what values to chose for q (or p resp.)?
- but: stationary problem $\Rightarrow$ independent of t
- thus: plot directions depending on p and q

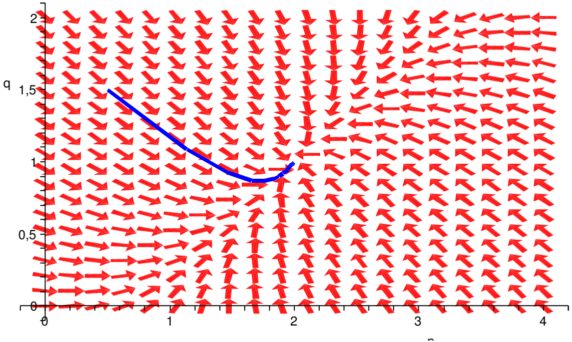
Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
Stability of Linear Systems
Stability of Non-Linear Systems

2D Direction Field – Arms Race

- system of differential equations:

$$\begin{aligned} \dot{p}(t) &= \tfrac{3}{2} - p(t) + \tfrac{1}{2}q(t) \\ \dot{q}(t) &= 0 + \tfrac{1}{2}p(t) - q(t) \end{aligned}$$

- direction field – with critical point at $(2, 1)$:

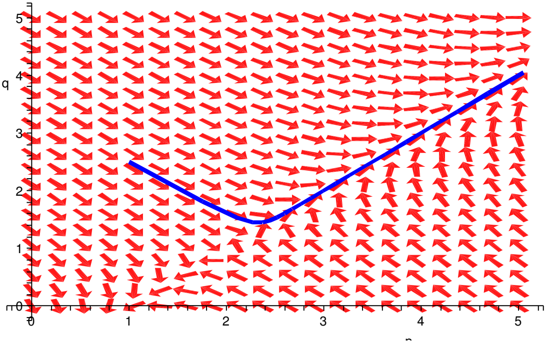


Arms Race – unlimited growth

- system of differential equations:

$$\begin{aligned} \dot{p}(t) &= \frac{1}{2} - \frac{3}{4}p(t) + q(t) \\ \dot{q}(t) &= -\frac{5}{4} + p(t) - \frac{3}{4}q(t) \end{aligned}$$

- direction field – with critical point at $(2, 1)$:

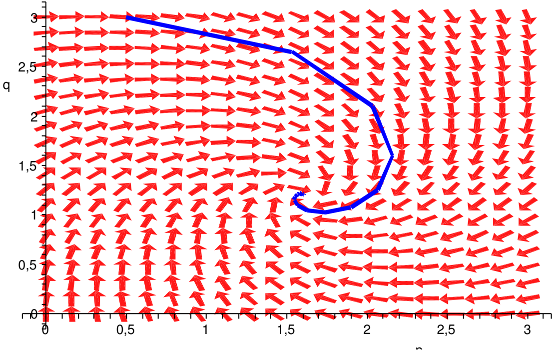


Arms race – the peaceful neighbour

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= 0 - \tfrac{3}{4}p(t) + q(t)\\ \dot{q}(t) &= \tfrac{5}{2} - p(t) - \tfrac{3}{4}q(t)\end{aligned}$$

- direction field – with critical point at $\left(\frac{8}{5}, \frac{6}{5}\right)$:

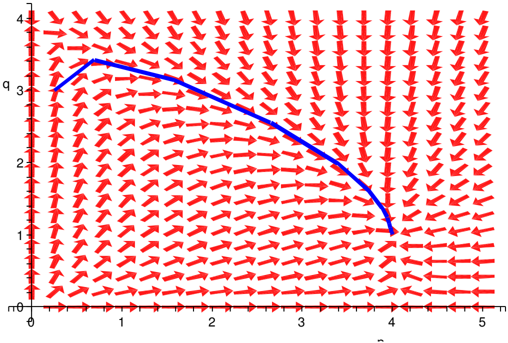


Nonlinear System – Competition

- system of differential equations:

$$\dot{p}(t) = \left(\frac{5}{2} + \frac{\sqrt{3}}{24} - \frac{5}{8}p(t) - \frac{\sqrt{3}}{24}q(t)\right)p(t)$$
$$\dot{q}(t) = \left(\frac{7}{8} + \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{8}p(t) - \frac{7}{8}q(t)\right)q(t)$$

- direction field – critical points at $(4,1),\ldots$:

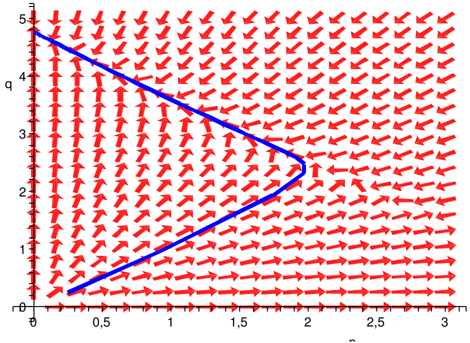


Nonlinear System – Extinction

- system of differential equations:

$$\dot{p}(t) = \left(\frac{71}{8} - \frac{23}{12}p(t) - \frac{25}{12}q(t)\right)p(t)$$
$$\dot{q}(t) = \left(\frac{73}{8} - \frac{25}{12}p(t) - \frac{23}{12}q(t)\right)q(t)$$

- critical points at $(0, 4.76\ldots), (4.63\ldots, 0), \ldots$:



Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
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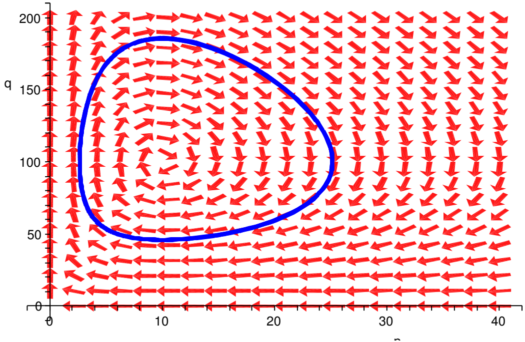
Lotka & Volterra

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= \left(-\tfrac{1}{2} + \tfrac{1}{200}q(t)\right)p(t)\\ \dot{q}(t) &= \left(\tfrac{1}{5} - \tfrac{1}{50}p(t)\right)q(t)\end{aligned}$$

- direction field – with critical point at $(10, 100)$:



2D Critical Points – Summary

Different types of critical points in 2D:

- attractive/stable equilibrium
  (arms race – steady state)
- unstable equilibrium
- saddle point (arms race – unlimited growth)
- attractive “spiral point” (“peaceful neighbour”)
- unstable “spiral point”
- centre of “rotation” (Lotka-Volterra)

$\Rightarrow$ How to discriminate between these types?

Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
Stability of Linear Systems
Stability of Non-Linear Systems

Homogeneous Systems of ODE

Homogeneous System in matrix-vector-notation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$
- example: $\mathbf{x}(t) = (p(t), q(t))$

Solutions:

- let $\mathbf{x}_\lambda$ be an eigenvector: $\mathbf{A}\mathbf{x}_\lambda = \lambda \mathbf{x}_\lambda$
- then $\mathbf{x}_\lambda e^{\lambda t}$ is a solution:

$$\mathbf{A}\mathbf{x}_\lambda e^{\lambda t} = \lambda \mathbf{x}_\lambda e^{\lambda t} = \frac{d}{dt}\left(\mathbf{x}_\lambda e^{\lambda t}\right) \qquad \text{q.e.d.}$$

Eigenvectors and Eigenvalues

Corollaries:

- the solutions of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are linear combinations of the respective eigen-solutions:

$$\mathbf{x}_{\mathbf{hom}}(t) = \sum_{\lambda} a_\lambda \mathbf{x}_\lambda e^{\lambda t}, \qquad a_\lambda \in \mathbb{R}$$

- the solutions of the inhomogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$ are

$$\mathbf{x}(t) = -\mathbf{A}^{-1} b + \mathbf{x}_{\mathbf{hom}}(t)$$

- observation: $\mathbf{x}_\mathbf{c} = -\mathbf{A}^{-1} b$ is a critical point!

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Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
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Eigenvalues and Critical Points

- the ODE system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$ is solved by

$$\mathbf{x}(t) = \mathbf{x_c} + \sum_{\lambda} a_\lambda \mathbf{x}_\lambda e^{\lambda t}$$

- $\mathbf{x_c}$ attractive equilibrium,

$$\lim_{x \longrightarrow \infty} \mathbf{x}(t) = \mathbf{x_c},$$

  only if $e^{\lambda t} \rightarrow 0$ for all eigenvalues λ
- $\lambda \in \mathbb{R} \Rightarrow \lambda < 0$
- $\lambda = \mu + i\nu \Rightarrow \mu < 0$ ($e^{i\nu t} = \cos \nu t + i \sin \nu t$)

Stability of Linear Systems

Overview:

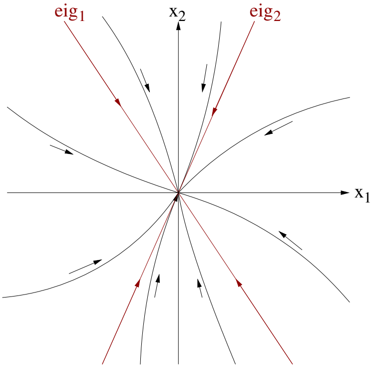
eigenval. ($\lambda_j = \mu_j + i\nu_j$)	critical point	stability
real, all $\lambda < 0$	node	stable, attr.
real, all $\lambda > 0$	node	unstable
real, $\lambda_k > 0, \lambda_l < 0$	saddle point	unstable
complex, all $\mu < 0$	spiral point	stable, attr.
complex, all $\mu > 0$	spiral point	unstable
complex, all $\mu = 0$	centre	stable

Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
Stability of Linear Systems
Stability of Non-Linear Systems

Stability of 2D Systems

Real Eigenvalues:

- $\lambda_1 < 0$, $\lambda_2 < 0$, attractive equilibrium



Critical Points

Points of Equilibrium

Critical Points

Direction Fields

Critical Points in 1D

Critical Points in 2D

2D Direction Fields

Summary

Analysis of Systems of ODE

Homogeneous Systems

Eigenvalues and Critical Points

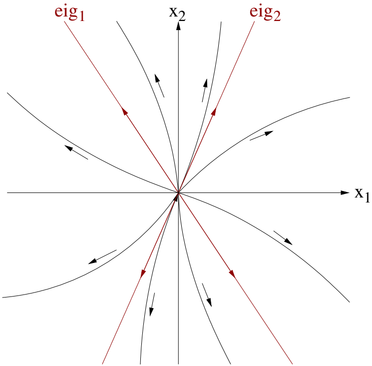
Stability of Linear Systems

Stability of Non-Linear Systems

Stability of 2D Systems

Real Eigenvalues:

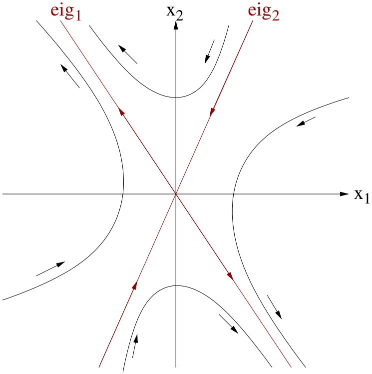
- $\lambda_1 > 0$, $\lambda_2 > 0$, unstable equilibrium



Stability of 2D Systems

Real Eigenvalues:

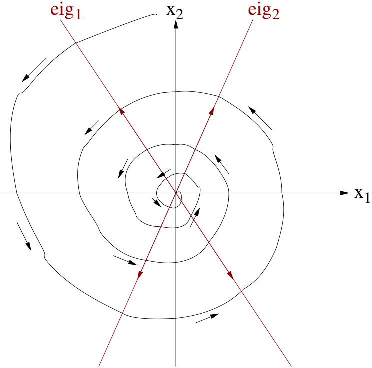
- $\lambda_1 > 0$, $\lambda_2 < 0$, saddle point



Stability of 2D Systems

Complex Eigenvalues:

- $\mu_1 < 0$, $\mu_2 < 0$, spiral point (asympt. stable)



Critical Points
Points of Equilibrium
Critical Points
Direction Fields
Critical Points in 1D
Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
Stability of Linear Systems
Stability of Non-Linear Systems

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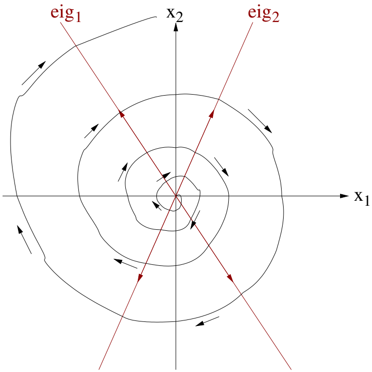
Complex Eigenvalues:

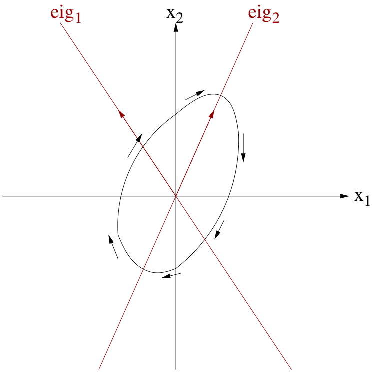
- $\mu_1 > 0$, $\mu_2 > 0$, spiral point (unstable)



Stability of 2D Systems

Complex Eigenvalues:

- $\mu_1 = \mu_2 = 0$, centre of oscillation



Stability of Non-Linear Systems

- 2D system of ODE:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)),$$

  $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ nonlinear
- critical point at $\mathbf{x_c}$: $f(\mathbf{x_c}(t)) = 0$
- for analysis of critical points: linearization

$$\dot{\mathbf{x}} = f(\mathbf{x}) \approx \underbrace{f(t, \mathbf{x_c})}_{=0} + \mathbf{J_f}(\mathbf{x_c})(\mathbf{x} - \mathbf{x_c})$$

- examine eigenvalues of $\mathbf{J_f}(\mathbf{x_c})$

Critical Points
Points of Equilibrium
Critical Points
Direction Fields
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Critical Points in 2D
2D Direction Fields
Summary
Analysis of Systems of ODE
Homogeneous Systems
Eigenvalues and Critical Points
Stability of Linear Systems
Stability of Non-Linear Systems