Technische Universität München
Institut für Informatik
Scientific Computing in Computer Science

Scientific Computing I

Final Exam – February 6, 2008

Name: ....

Matr.Nr.: ....

Program: ....

General Instructions

Material: You may only use one hand-written sheet of paper (size A4, on both pages). All other material including electronic devices of any kind are forbidden.

You may use this exercise sheet and the exam paper that was handed out to solve the exercises (for notes and sketches, you can obtain additional exam sheets).

Do not use pencil, or red or green ink.

General hint: Often, exercises b), c), etc. can be solved without the results from the previous exercise a): if you are stuck with exercise a, then don't immediately skip exercises b, c, etc.

Maximum score: The maximum score is 40 points plus a bonus of 8 points. Grades will be computed relative to a maximum score of 40 points.

17 points are required to pass the exam.

Working time: 90 minutes.

1 Direction Fields for ODE (≈ 7 points)

Consider the following ODE:

$$\dot{p}(t) = c\left(1 - \frac{p(t)}{a}\right)\left(1 - \frac{p(t)}{b}\right)p(t) \tag{1}$$

Give the critical points of this ODE and draw a rough sketch of the direction field of this PDE (for the case $a, b > 0$, $c < 0$, and $a < b$). Add three typical solutions to your sketch – the solutions should reflect three different scenarios of how the modelled population can evolve. State the name of the population model that uses this ODE.

2 Numerical Methods for ODE (≈ 4 + 4 = 8 points)

Consider the direction field of a 1D ordinary differential equation $\dot{p}(t) = f(t, p(t))$, as given in figure 1.

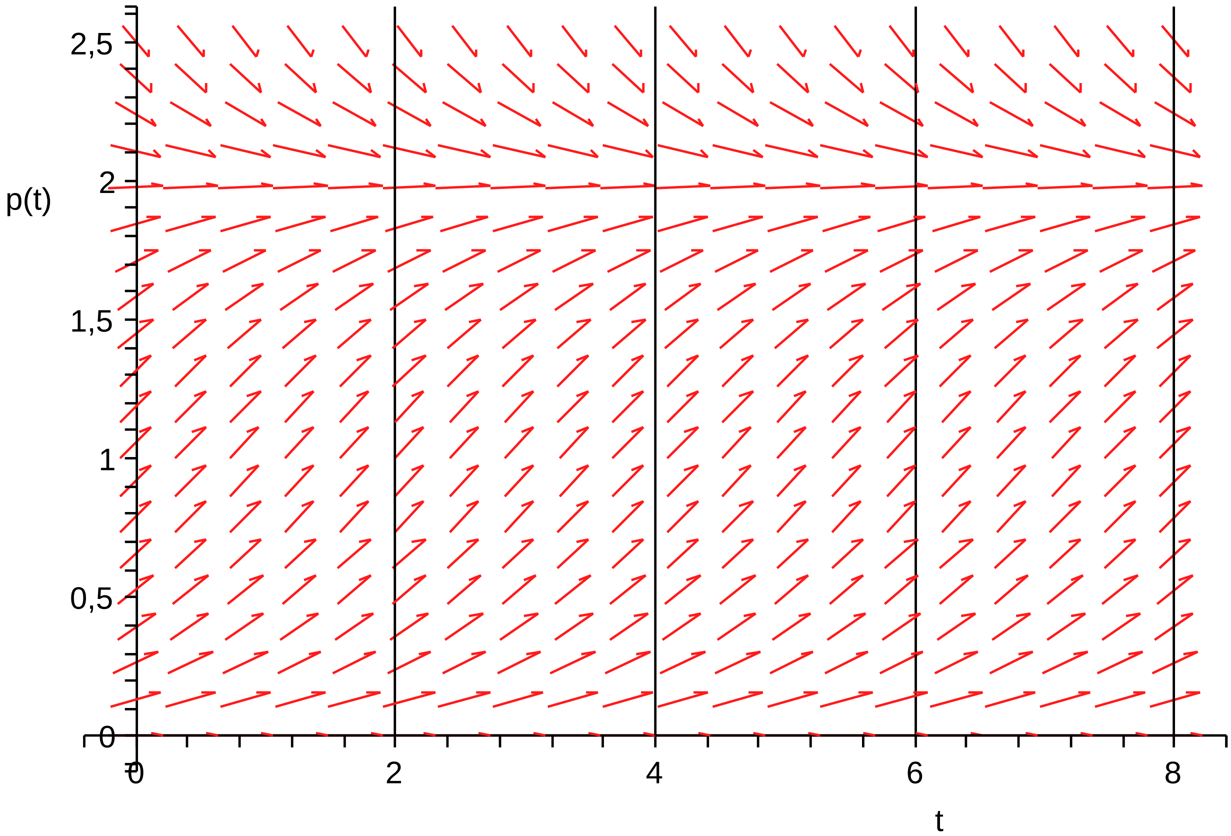


Figure 1: Direction field for exercise 2 – draw the solutions of exercises 2a directly into this diagram.

a) To compute approximate solutions $p_n \approx p(t_n)$ at times $t_n = n \cdot \tau$, the following numerical scheme is given:

$$p_0 = p(0) = \frac{1}{8} \qquad \text{(initial condition)} \tag{2}$$

$$p_1 = p_0 + \tau \cdot f(t_0, p_0) \tag{3}$$

$$p_{n+1} = p_{n-1} + 2\tau \cdot f(t_n, p_n) \qquad \text{for } n = 1, 2, 3, \ldots. \tag{4}$$

Perform the first four steps of this scheme (to compute p_1, p_2, p_3, p_4) by drawing the approximate solutions into the direction field in figure 1 (graphical solution only).
The stepsize shall be $\tau = 2$, as illustrated by the four intervals drawn into the direction field. Mark from which arrows you obtain the directions of the numerical steps – you are allowed to add an arrow to the direction field, if no arrow is plotted at the precise required position.

b) Specify a second-order Runge-Kutta method to compute the approximation for p_1. Draw a sketch (analogous to exercise a) that illustrates how the respective numerical solution is obtained for this scheme. Why should the step to compute p_1 be replaced by the Runge-Kutta scheme?

3 Population Modelling (≈ 5 + 2 + 2 = 9 points)

Given are four different scenarios of a two-species population model. Figure 2 shows the direction fields for these four scenarios. Figure 3 shows the respective solution plots.

a) State which solution plot belongs to which direction field. Give a short statement about your reasons for the choice. Draw the evolution of the population sizes p and q into the four plots in figure 2.

b) One of the four scenarios uses a non-linear model; the other three scenarios are from a linear model. State which solution plot and direction field belong to the non-linear model and give a short statement about the reason for your choice.

c) All four scenarios have a critical point for the same values of p and q. State what type of critical point it is for each of the models.

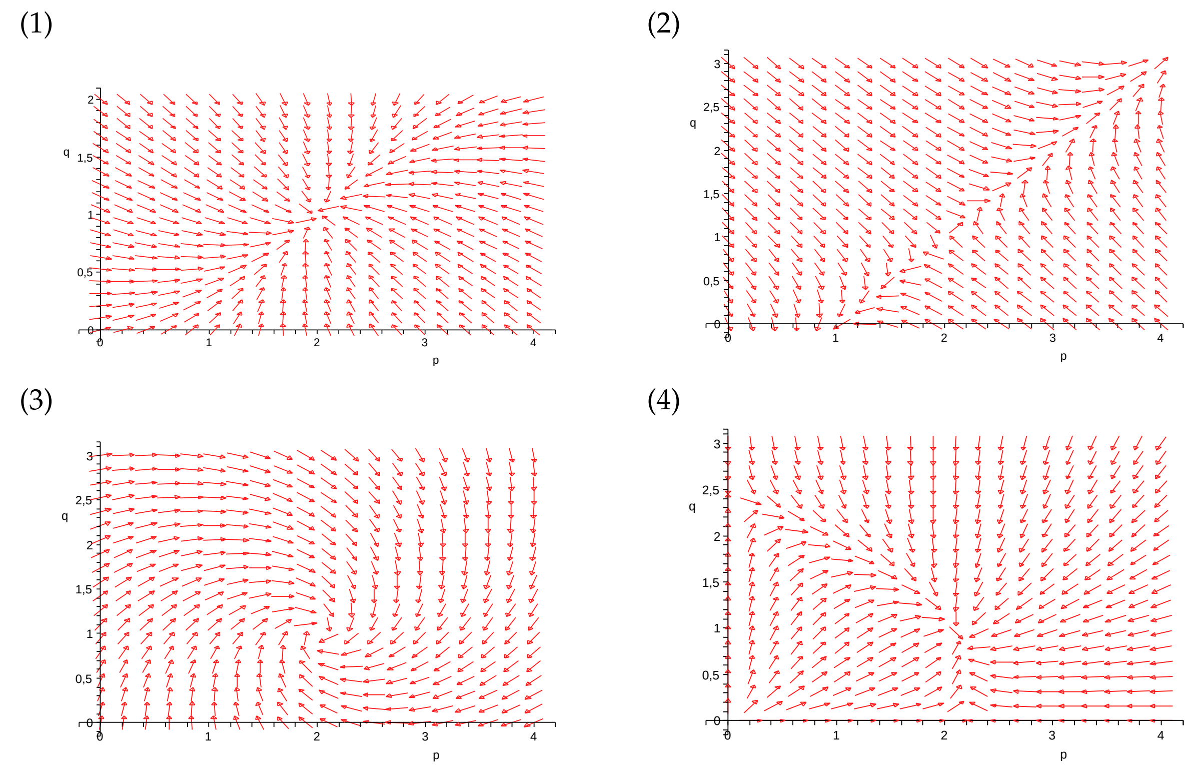


Figure 2: Direction fields for exercise 3. Population p is plotted along the horizontal axis, population q is plotted along the vertical axis.

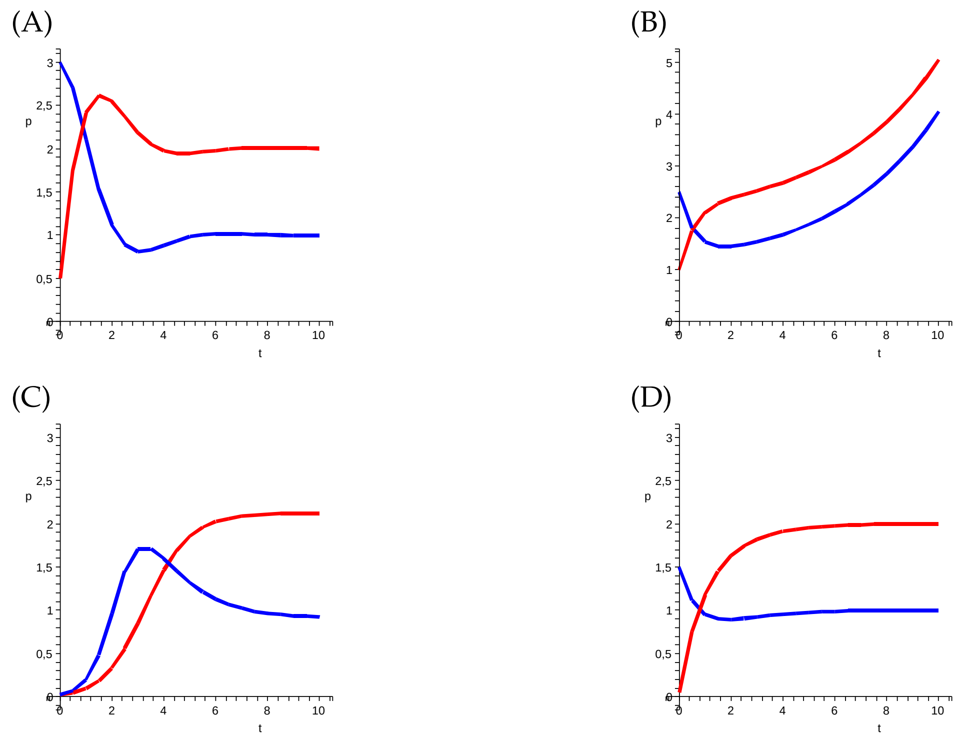


Figure 3: Solution plots for exercise 3. Population p is plotted in red, population q is plotted in blue.

4 Numerics for PDE, Neumann Stability (≈ 5 + 2 + 2 = 9 points)

The discretisation of the 1D partial differential equation $u_t = u - u_{xx}$ using the forward Euler method leads to the following numerical scheme:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = u_j^{(m)} - \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{h^2} \tag{5}$$

with Dirichlet boundary conditions: $u_0^{(m)} = u_n^{(m)} = 0$ for all m.

As usual, we denote $u_j^{(m)} \approx u(t_m, x_j)$ with $t_m = m \cdot \tau$ and $x_j = j \cdot h$. Hence, in the approximate solution $u_j^{(m)}$, the superscript (m) denotes the time step, and the index j denotes the grid point.

a) Similar to the lectures, we assume that equation (5) has solutions of the form

$$u_j^{(m)} := (a_k)^m \sin(k\pi x_j)$$

for different values of k. Compute the respective values of the a_k.

Hint: *Use that* $x_{j\pm1} = x_j \pm h$, *and that* $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$.

b) Compute α, such that $u(x,t) = e^{\alpha t}\sin(k\pi x)$ is a solution of the PDE $u_t = u - u_{xx}$.

c) Considering the results of (a) and (b), do the numerical solutions reflect the overall behaviour of the exact solution? (If they only do under specific conditions, then specify these conditions.)

5 Finite Elements (≈ 3 + 7 = 10 points)

Consider the 1D-problem

$$u(x) - \frac{\partial^2}{\partial x^2} u(x) = f(x) \tag{6}$$

on the unit interval $\Omega = (0,1)$ with homgeneous Dirichlet boundary conditions: $u(0) = u(1) = 0$.

a) Give a weak formulation of equation (6).

b) For the Finite Element discretisation, use the equidistant grid points $x_i := i/n$ for $i = 1, \ldots, n-1$, and the nodal basis as presented in the lectures:

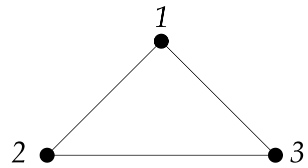
$$\varphi_i(x) := \begin{cases} \frac{1}{h}(x - x_{i-1}) & x_{i-1} < x \le x_i \\ \frac{1}{h}(x_{i+1} - x) & x_i < x < x_{i+1} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Compute the discretisation stencil of the resulting Finite Element discretisation (using the $\varphi_i, i = 1, \ldots, n-1$ as test and ansatz functions) of equation (6).

You may "compute" the value of integrals in a graphical way, just as we did in the lectures. You do not need to recompute values that can be derived from symmetries.

6 Element Stiffness Matrices ($\approx$ 5 points)

Assume that the Finite Element discretisation of a certain PDE uses right-angled triangular elements, i.e.:

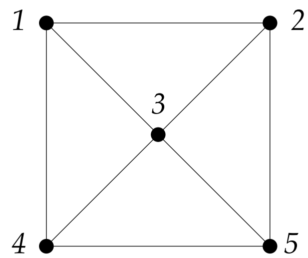


With certain nodal basis and test functions, you obtain the element stiffness matrix

$$\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

which in the given example shall be valid for all elements regardless of their orientation and size. The numbering of the unknowns and their location within the elements is given in the above picture: unknown 1 always sits on the node with the right angle.

The entire computational grid shall consist of 4 such triangles (with five unknowns, altogether):



Set up the global stiffness matrix for this computational grid. Be careful to use the given numbering of the five unknowns. Use only the given local stiffness matrix (i.e. do not consider any boundary values or similar).