Scientific Computing I
Module 5: Heat Transfer – Discrete and Continuous Models

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Part I

Discrete Models
Motivation: Heat Transfer

- **objective:** compute the temperature distribution of some object
- **under certain prerequisites:**
  - temperature at object boundaries given
  - heat sources
  - material parameters
- **observation from physical experiments:**

\[ q \approx k \cdot \delta T \]

heat flow proportional to temperature differences
A Wiremesh Model

- consider rectangular plate as fine mesh of wires
- compute temperature $x_{ij}$ at nodes of the mesh
Wiremesh Model (2)

- model assumption: temperatures in equilibrium at every mesh node
- for all temperatures $x_{ij}$:
  \[ x_{ij} = \frac{1}{4} \left( x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1} \right) \]
- temperature known at (part of) the boundary; for example:
  \[ x_{0,j} = T_j \]
- task: solve system of linear equations
A Finite Volume Model

- object: a rectangular metal plate (again)
- model as a collection of small connected rectangular cells

examine the heat flow across the cell edges
Heat Flow Across the Cell Boundaries

- Heat flow across a given edge is proportional to
  - temperature difference \((T_1 - T_0)\) between the adjacent cells
  - length \(h\) of the edge
- e.g.: heat flow across the left edge:

\[
q_{ij}^{\text{(left)}} = k_x (T_{ij} - T_{i-1,j}) h_y
\]

- heat flow across all edges determines change of heat energy:

\[
q_{ij} = k_x (T_{ij} - T_{i-1,j}) h_y + k_x (T_{ij} - T_{i+1,j}) h_y
+ k_y (T_{ij} - T_{i,j-1}) h_x + k_y (T_{ij} - T_{i,j+1}) h_x
\]
Temperature change due to heat flow

- in equilibrium: total heat flow equal to 0
- but: consider additional source term $F_{ij}$ due to
  - external heating
  - radiation
- $F_{ij} = f_{ij} h_x h_y$ ($f_{ij}$ heat flow per area)
- equilibrium with source term requires
  $q_{ij} + F_{ij} = 0$:

$$f_{ij} h_x h_y = -k_x h_y \left( 2T_{ij} - T_{i-1,j} - T_{i+1,j} \right)$$
$$-k_y h_x \left( 2T_{ij} - T_{i,j-1} - T_{i,j+1} \right)$$
Finite Volume Model

- divide by $h_x h_y$:

$$f_{ij} = -\frac{k_x}{h_x} \left(2T_{ij} - T_{i-1,j} - T_{i+1,j}\right) \quad \text{and} \quad -\frac{k_y}{h_y} \left(2T_{ij} - T_{i,j-1} - T_{i,j+1}\right)$$

- again, system of linear equations
- how to treat boundaries?
  - prescribe temperature in a cell (e.g. boundary layer of cells)
  - prescribe heat flow across an edge; for example insulation at left edge:
    $$q_{ij}^{(\text{left})} = 0$$
Towards a Time Dependent Model

- idea: set up ODE for each cell
- simplification: no external heat sources or drains, i.e. $f_{ij} = 0$
- change of temperature per time is proportional to heat flow into the cell (no longer 0):

$$\dot{T}_{ij}(t) = \frac{\kappa_x}{h_x} (2T_{ij}(t) - T_{i-1,j}(t) - T_{i+1,j}(t))$$

$$+ \frac{\kappa_y}{h_y} (2T_{ij}(t) - T_{i,j-1}(t) - T_{i,j+1}(t))$$

- solve system of ODE
Part II

A Continuous Model – The Heat Equation
From Discrete to Continuous

- remember the discrete model:

\[ f_{ij} = -\frac{k_x}{h_x} (2T_{ij} - T_{i-1,j} - T_{i+1,j}) - \frac{k_y}{h_y} (2T_{ij} - T_{i,j-1} - T_{i,j+1}) \]

- assumption: heat flow across edges is proportional to temperature difference

\[ q_{ij}^{(left)} = k_x (T_{ij} - T_{i-1,j}) h_y \]

- in reality: heat flow proportional to temperature gradient

\[ q_{ij}^{(left)} \approx kh_y \frac{T_{ij} - T_{i-1,j}}{h_x} \]
From Discrete to Continuous (2)

- replace $k_x$ by $k/h_x$, $k_y$ by $k/h_y$, and get:

$$f_{ij} = -\frac{k}{h_x^2} (2T_{ij} - T_{i-1,j} - T_{i+1,j})$$
$$-\frac{k}{h_y^2} (2T_{ij} - T_{i,j-1} - T_{i,j+1})$$

- consider arbitrary small cells: $h_x, h_y \to 0$:

$$f_{ij} = -k \left( \frac{\partial^2 T}{\partial x^2} \right)_{ij} - k \left( \frac{\partial^2 T}{\partial y^2} \right)_{ij}$$

- leads to partial differential equation (PDE):

$$-k \left( \frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} \right) = f(x,y)$$
Derivation of the Heat Equation

- finite volume model, but with arbitrary control volume $D$
- change of heat energy (per time) is a result of
  - transfer of heat energy across $D$’s surface,
  - heat sources and drains in $D$ (external influences)
- resulting integral equation:

\[
\frac{\partial}{\partial t} \int_D \rho c T \, dV = \int_D q \, dV + \int_{\partial D} k \nabla T \cdot \vec{n} \, dS
\]

- density $\rho$, specific heat $c$, and heat conductivity $k$ are material parameters
- heat sources and drains are modelled in term $q$
Derivation of the Heat Equation (2)

- according to theorem of Gauß:
  \[ \int_{\partial D} k \nabla T \cdot \vec{n} \, dS = \int_D k \Delta T \, dV \]

- leads to integral equation for any domain \( D \):
  \[ \int_D \rho c T_t - q - k \Delta T \, dV = 0 \]

- hence, the integrand has to be identically 0:
  \[ T_t = \kappa \Delta T + \frac{q}{\rho c}, \quad \kappa := \frac{k}{\rho c} \]

- \( \kappa > 0 \) is called the thermal diffusion coefficient (since the Laplace operator models a (heat) diffusion process)
Heat Equations

Different scenarios:

- vanishing external influence, \( q = 0 \):
  \[
  T_t = \kappa \Delta T
  \]

  alternate notation
  \[
  \frac{\partial T}{\partial t} = \kappa \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)
  \]

- equilibrium solution, \( T_t = 0 \):
  \[
  0 = \kappa \Delta T + \frac{q}{\rho c} \quad \longrightarrow \quad -\Delta T = f
  \]

“Poisson’s Equation”
Boundary Conditions

Dirichlet boundary conditions:
- fix $T$ on (part of) the boundary

$$T(x,y,z) = \varphi(x,y,z)$$

Neumann boundary conditions:
- fix $T$’s normal derivative on (part of) the boundary:

$$\frac{\partial T}{\partial n}(x,y,z) = \varphi(x,y,z)$$

- special case: insulation

$$\frac{\partial T}{\partial n}(x,y,z) = 0$$