Introduction to Scientific Computing

Neumann Stability Analysis

We examine the 1D heat equation,

$$u_t(x,t) = u_{xx}(x,t) \qquad \text{in} \qquad (0,1),$$

with suitable initial and boundary conditions $u(0,t) = u(1,t) = 0$.

To compute a numerical solution, we have derived the explicit scheme

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = \frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{h^2} \tag{1}$$

Exercise 1

We assume that the discrete equation (1) has solutions similar to that of the continuous heat equation, i.e. that is has solutions of the type

$$u_j^{(m)} = (a_k)^m \sin(\pi k x_j), \qquad \text{where } x_j := jh. \tag{2}$$

Note that the term $(a_k)^m$ is similar to an exponential function with negative exponent, if $0 < a_k < 1$.

Compute the a_k such that the (presumed) solutions given in equation (2) satisfy equation (1).

Hint: You will most likely need to use the theorem $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$ *during your computation.*

Exercise 2

For each k, exercise 1 leads us to a possible solution of the discretized heat equation. Specify the respective initial conditions for these solutions.

Assume, we have a given set of initial values f_j:

$$u_j^{(0)} := f_j \qquad \text{for } j = 1, \ldots, n-1,$$

Try to write the f_j as a superposition of initial conditions for which you know the solution. State the solution of the discretized heat equation with the f_j as initial conditions.

Exercise 3

Now, the solution of the discretized equation (1) shall be given as a superposition of functions given in equation (2), i.e.

$$v_j^{(m)} := \sum_{j=0}^{n} c_k (a_k)^m \sin(\pi k x_j) \tag{3}$$

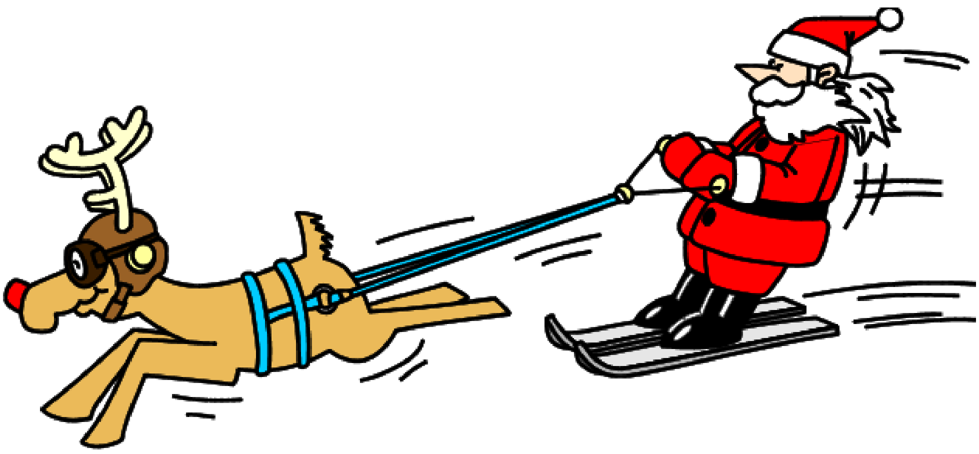
for given c_k, and the a_k computed as in exercise 1.

Characterize the solution $v_j^{(m)}$ (esp. for $m \to \infty$), if

- $|a_k| < 1$ for all k;
- $|a_k| > 1$ for at least one value of k.

Give a sufficient condition to make sure that $|a_k| < 1$ for all k.

Hint: use that $\sin^2(A) = \frac{1}{2}(1 - \cos(2A))$.



Merry Christmas and all the best for the new year!