Scientific Computing I

Solutions Exam WS07

1 Direction Fields for ODE

Critical points:
\[ \dot{p} = 0 \Leftrightarrow p = 0 \text{ or } p = a \text{ or } p = b \]

Sketch of the direction field:

\[
\begin{align*}
0 \leq p < a & \Rightarrow \dot{p} < 0, \\
a < p < b & \Rightarrow \dot{p} > 0, \\
p > b & \Rightarrow \dot{p} < 0
\end{align*}
\]

The Model is called logistic growth with threshold.

2 Numerical Methods for ODE

a) blue lines in the sketch

b) Second order Runge-Kutta:

\[
p_{n+1} = p_n + \frac{\tau}{2} \left( f(t_n, p_n) + f(t_{n+1}, y_n + \tau f(t_n, y_n)) \right).
\]
Replacing the first step in a) by the Runge-Kutta scheme leads to an 2nd order approximation accuracy in all time steps, whereas with the explicit Euler for $p_1$, the first step is only first order accurate.

3 Population Modelling

a) (B) belongs to (2) (no stable critical point in the direction field, but simulatneous growth of both, $p$ and $q$ for large $t$).

(A) belongs to (3) (stable critical point, i.e. equlilbrum for $t \to \infty$ at $p = 2$ and $q = 1$, change of sign in the slope of $p$ and $q$ due to spiral point).

(C) belongs to (4) (critical point at equilibrium of the solution plots, not-monotonic behaviour of $q$ if starting from $(0,0)$).

(D) belongs to (1) (critical point at equilibrium of the solution plots, monotonic growth of $p$ and decrease of $q$ towards equil. if starting from $(0,1.5)$).
b) (4) belongs to a non-linear model as the two eigenvalues of the system obviously depend on $p$ and $q$.

c) (1) stable equilibrium (no spiral form of direction field, arrows from all directions pointing towards the critical point)
(2) saddle point (attracting solutions only along a line, unstable in another line direction)
(3) stable/attractive spiral point (spiral form of direction fields, all arrows pointing towards the critical point)
(4) stable equilibrium (see (1)).

4 Numerics for PDE, Neumann Stability

a) Insert the specific form of the solution to (5):

\[
\begin{align*}
\frac{(a_k)^{m+1} - (a_k)^m \sin(k\pi x_j)}{\tau} & = (a_k)^m \left[ \frac{\sin(k\pi x_j) - \sin(k\pi x_{j+1}) - 2\sin(k\pi x_j) + \sin(k\pi x_{j-1})}{h^2} \right], \\
\frac{a_k - 1}{\tau} & = 1 - \frac{2\cos(k\pi h) - 2}{h^2}, \\
a_k & = 1 + \tau - 2\frac{\cos(k\pi h) - 1}{h^2}.
\end{align*}
\]

b) Insert the specific form of the solution to the PDE:
\[\alpha e^{\alpha t} \sin(k\pi x) = e^{\alpha t} \sin(k\pi x) + (k\pi)^2 e^{\alpha t} \sin(k\pi x),\]

\[\alpha = 1 + (k\pi)^2.\]

c) From a), we get \(a_k > 1\) as \(\cos(k\pi h) - 1 \leq 0\), thus the amplitude of \(u_k^{(m)}\) grows to \(\infty\).

b) gives \(\alpha > 1\). Thus, the amplitude of \(u\) also grows to \(\infty\).

The numerical solutions reflects the overall behaviour of the exact solution.

5 Finite Elements

a) \(\int_0^1 v(x)u(x)\,dx - \int_0^1 v(x)u''(x)\,dx = \int_0^1 v(x)f(x).\)

Apply Green’s formula and use zero boundary conditions:
\(\int_0^1 v(x)u(x)\,dx + \int_0^1 v'(x)u'(x)\,dx = \int_0^1 v(x)f(x).\)

b)

\[A_{i,i} = \int_0^1 \phi_i \phi_i + \phi_i' \phi_i' \,dx =\]

\[2 \cdot \int_{x_{i-1}}^{x_i} \frac{1}{h^2} (x - x_{i-1})^2 \,dx + 2h \cdot \frac{1}{h^2} =\]

\[2 \cdot \int_{x_{i-1}}^{x_i} x^2 \,dx + \frac{2}{h} =\]

\[\frac{2}{3}h + \frac{2}{h},\]

\[A_{i,i-1} = \frac{1}{h^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})(x_i - x) - \frac{1}{h^2} =\]

\[\frac{1}{h^2} \int_0^h x(h - x)\,dx - \frac{1}{h} =\]

\[\frac{1}{h^2} \left( h \frac{h^2}{2} - \frac{h^3}{3} \right) - \frac{1}{h} =\]

\[\frac{h}{6} - \frac{1}{h},\]

\[A_{i,i+1} = A_{i,i-1} \text{ (symmetry).}\]

Stencil: \([\frac{h}{6} - \frac{1}{h} \quad \frac{2}{3}h + \frac{2}{h} \quad h - \frac{1}{h}].\)

6 Element Stiffness Matrices

In a first step, we transform the local element stiffness matrices to the size of the global system, i.e. reorder entries and fill in zeros where necessary to get from the local ordering to the global ordering of unknowns:
• lower triangle in the global system:
Local number one becomes global number three, local number two global number 4, and local number 3 global number 5. Thus, we have to enhance the matrix by two zero rows and columns corresponding to the (global) unknown numbers one and two:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\]

• upper triangle in the global system:
Local number one becomes global number three, local number two global number 2, and local number 3 global number 1. Thus, we have to enhance the matrix by two zero rows and columns corresponding to the (global) unknown numbers 4 and five and reorder the first three rows:

\[
\begin{pmatrix}
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

• left triangle in the global system:
Local number one becomes global number three, local number two global number 1, and local number 3 global number 4. Thus, we have to enhance the matrix by two zero rows and columns corresponding to the (global) unknown numbers 2 and five and reorder the other rows:

\[
\begin{pmatrix}
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

• right triangle in the global system:
Local number one becomes global number three, local number two global number 5, and local number 3 global number 2. Thus, we have to enhance the matrix by two zero rows and columns corresponding to the (global) unknown numbers 1 and four and reorder the other rows:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\]
We add all these matrices and get the global system matrix:

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 & 1
\end{pmatrix}.
\]