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WT 2004/2005

December 22, 2004

Introduction to Scientific Computing

Final Exam, February 9th 2005

1 Population Dynamics

a) Critical points:

$\dot{p} = -r\left(1 - \frac{p}{k}\right)\left(1 - \frac{p}{l}\right) = 0 \Leftrightarrow p = k \text{ or } p = l$

Stability of the critical points:

$\frac{pf}{dp}(p) = -r\left(1 - \frac{p}{k}\right)\left(-\frac{1}{l}\right) - r\left(-\frac{1}{k}\right)\left(1 - \frac{p}{k}\right) = r\frac{l+k-2p}{kl}.$

$\frac{df}{dp}(l) = r\frac{l+k-2l}{kl} > 0 \Rightarrow$ unstable critical point at $p = l$.

$\frac{df}{dp}(k) = r\frac{l+k-2k}{kl} < 0 \Rightarrow$ stable critical point at $p = k$.

b) Limit for $t \to \infty$:

I) smallest critical point at $p = l$, unstable
$\Rightarrow \lim_{t\to\infty} p(t) = -\infty$ for $0 \le p_0 < l$,

II) critical point at $p = l$
$\Rightarrow \lim_{t\to\infty} p(t) = l$ for $p_0 = l$,

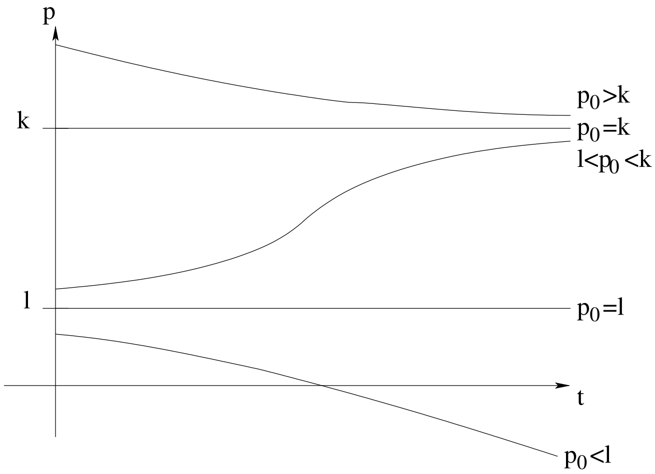
III) unstable critical point at $p = l$, stable critical point at $p = k$, no critical points in between
$\Rightarrow \lim_{t\to\infty} p(t) = k$ for $l < p_0 < k$,

IV) critical point at $p = k$
$\Rightarrow \lim_{t\to\infty} p(t) = k$ for $p_0 = k$.

V) biggest critical point at $p = k$, stable
$\Rightarrow \lim_{t\to\infty} p(t) = k$ for $p_0 > k$.

c) $\frac{df}{dp}(p) = r\frac{l+k-2p}{kl} = 0 \Leftrightarrow p = \frac{l+k}{2}.$

d)



e) $\lim_{t\to\infty} p(t) = -\infty$ for $p_0 < l$.

2 Explicit Midpoint Rule

a) Discrete equation:

$$\frac{y_{k+1}-y_k}{\delta t} = f\left(t_k + \frac{\delta t}{2}, y_k + \frac{\delta t}{2} f(t_k, y_k)\right).$$

insert the exact solution $y(t)$:

left hand side:

$$\frac{y(t_{k+1}) - y(t_k}{\delta t} = \frac{y(t_k) + \delta t \cdot f(t_k, y_k) + \frac{\delta t^2}{2}\frac{d^2 f}{dt^2}(t_k, y_k) + O(\delta t^3) - y(t_k)}{\delta t} =$$

$$f(t_k, y(t_k)) + \frac{\delta t}{2}\frac{df}{dt}(t_k, y(t_k)) + O(\delta t^2).$$

right hand side:

$$f\left(t_k + \frac{\delta t}{2}, y(t_k) + \frac{\delta t}{2} f(t_k, y(t_k))\right) =$$

$$f(t_k, y(t_k)) + \frac{\delta t}{2} f_t(t_k, y(t_k)) + \frac{\delta t}{2} f(t_k, y(t_k)) f_y(t_k, y(t_k)) + O(\delta t^2) =$$

$$f(t_k, y(t_k)) + \frac{\delta t}{2}\frac{df}{dt}(t_k, y(t_k)) + O(\delta t^2).$$

$\Rightarrow$ left hand side $-$ right hand side $= O(\delta t^2)$.

b) The midpoint rule is a one-step method, thus, it is convergent and the order of convergence equals the order of consistency. Therefore, the midpoint rule is at least second order convergent.

c) $y_k = y_{k-1} + \delta t \cdot \lambda \cdot \left(y_{k-1} + \frac{\delta t}{2} \cdot \lambda \cdot y_{k-1}\right) = \left(1 + \lambda \delta t + \frac{\lambda^2}{2}\delta t^2\right) y_{k-1}$

$$= \ldots = \left(1 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2\right)^k y_0.$$

$$\Rightarrow |y_k| \leq |y_0| \Leftrightarrow \left|1 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2\right| \leq 1 \Leftrightarrow -1 \leq 1 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2 \leq 1.$$

- $-1 \leq 1 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2$

  $\Leftrightarrow 0 \leq 2 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2 = \left(\frac{\lambda}{\sqrt{2}}\delta t + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}$

  $\Rightarrow$ no restrictions for δt from this case
- $1 + \lambda\delta t + \frac{\lambda^2}{2}\delta t^2 \leq 1$

  $\Leftrightarrow \lambda\delta t + \frac{\lambda^2}{2}\delta t^2 = \underbrace{\lambda\delta t}_{<0}\left(1 + \frac{\lambda}{2}\delta t\right) \leq 0$

  $\Leftrightarrow 1 + \frac{\lambda}{2}\delta t \geq 0 \Leftrightarrow -\frac{\lambda}{2}\delta t \leq 1$

  $\Leftrightarrow \delta t \leq -\frac{2}{\lambda}$

$\Rightarrow$ The stability condition is fulfilled for $\delta t \leq -\frac{2}{\lambda}$.