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Introduction to Scientific Computing

Midterm – Solution

1 Convergence of the Midpoint Rule

a) Scaled midpoint rule: $\frac{y_{k+1} - y_{k-1}}{2h} = f(x_k, y_k)$.

Insert the exact solution $y(x)$ of the ODE:

$\frac{y(x_{k+1}) - y(x_{k-1})}{2h} - f(x_k, y(x_k)) =$

$\frac{y(x_k) + h \cdot y'(x_k) + \frac{h^2}{2} \cdot y''(x_k) + \frac{h^3}{6} \cdot y'''(x_k) + O(h^4) - \Big[y(x_k) - h \cdot y'(x_k) + \frac{h^2}{2} \cdot y''(x_k) - \frac{h^3}{6} \cdot y'''(x_k) + O(}{2h}$

$f(x_k, y(x_k)) =$

$y'(x_k) + \frac{h^2}{6} \cdot y'''(x_k) - f(x_k, y(x_k)) + O(h^3) =$

$\frac{h^2}{6} \cdot y'''(x_k) + O(h^3) = O(h^2)$.

b) associated polynomial:

$\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$.

The associated polynomial has two roots $\lambda_1 = 1, \lambda_2 = -1$, which both have the absolute value one and are simple roots.

Thus, the midpoint rule is stable.

c) consistency + stability = convergence

order of convergence = order of consistency

Thus, the midpoint rule is second order convergent.

2 Two-Dimensional Systems of Linear First Order Ordinary Differential Equations

a) eigenvalues of the system matrix:

$$\begin{vmatrix} -3-\lambda & 14 \\ 4 & -2-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda) - 56 = \lambda^2 + 5\lambda - 50.$$

$$\Rightarrow \lambda_{1/2} = \frac{-5 \pm \sqrt{25+200}}{2} = \frac{-5 \pm 15}{2}$$

$\lambda_1 = -10, \quad \lambda_2 = 5.$

eigenvectors:

- $\lambda_1 = -10$:

$$\begin{pmatrix} -3+10 & 14 \\ 4 & -2+10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 7 & 14 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

- $\lambda_2 = 5$:

$$\begin{pmatrix} -3-5 & 14 \\ 4 & -2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

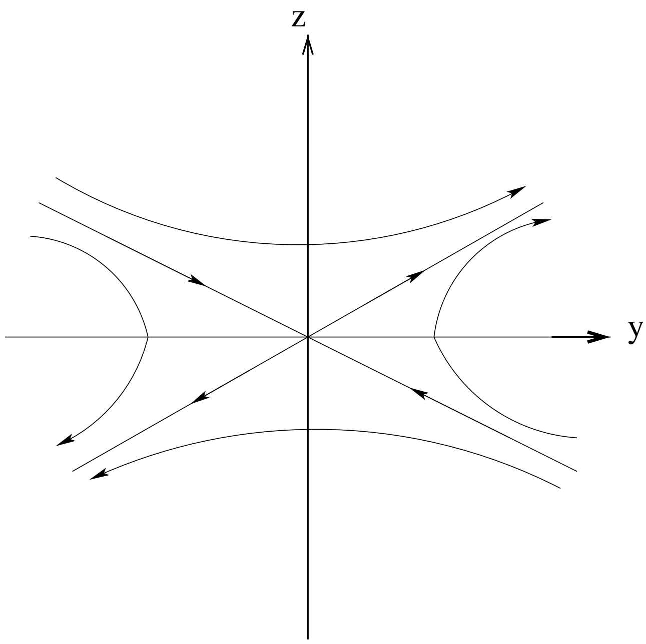
$$\Leftrightarrow \begin{pmatrix} -8 & 14 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b \cdot \begin{pmatrix} 7 \\ 4 \end{pmatrix}.$$

general solution:

$$\begin{pmatrix} y \\ z \end{pmatrix}(x) = a \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-10x} + b \cdot \begin{pmatrix} 7 \\ 4 \end{pmatrix} e^{5x}.$$

Type of the critical point $(0,0)$: $\lambda_1 < 0, \lambda_2 > 0$

$\Rightarrow$ saddle point, unstable



b) general form of y: $y(x) = \bar{a} \cdot e^{-10x} + \bar{b} \cdot e^{5x}$

I)
$$\left.\begin{aligned} y_i = e \quad &\Leftrightarrow \quad \bar{a} \cdot e + \bar{b} \cdot e^{-0.5} = e \\ y_i' = -10e \quad &\Leftrightarrow \quad -10\bar{a} \cdot e + 5\bar{b} \cdot e^{-0.5} = -10e \end{aligned}\right\} \Rightarrow 15\bar{b} \cdot e^{-0.5} = 0$$

$\Rightarrow \bar{b} = 0, \quad \bar{a} = 1$

$\Rightarrow y(x) = e^{-10x}$.

$$\text{II)} \quad \left.\begin{array}{lll} y_i = e + \epsilon & \Leftrightarrow & \bar{a} \cdot e + \bar{b} \cdot e^{-0.5} = e + \epsilon \\ y_i' = -10e & \Leftrightarrow & -10\bar{a} \cdot e + 5\bar{b} \cdot e^{-0.5} = -10e \end{array}\right\} \Rightarrow 15\bar{b} \cdot e^{-0.5} = 10\epsilon$$

$\Rightarrow \bar{b} = \frac{2}{3}e^{0.5} \cdot \epsilon, \quad \bar{a} = e^{-1}(e + \epsilon - \frac{2}{3}\epsilon) = 1 + \frac{1}{3}\epsilon \cdot e^{-1}$.

$\Rightarrow y_\epsilon(x) = \left(1 + \frac{1}{3}\epsilon \cdot e^{-1}\right) e^{-10x} + \frac{2}{3}\epsilon \cdot e^{5x+0.5}$.

c) $e(\epsilon) = \dfrac{y(x) - y_\epsilon(x)}{y(x)} = \dfrac{-\frac{1}{3}\epsilon \cdot e^{-1-10x} - \frac{2}{3}\epsilon \cdot e^{5x+0.5}}{e^{-10x}} =$
$-\dfrac{1}{3}\epsilon \cdot e^{-1} - \dfrac{2}{3}\epsilon \cdot e^{15x+0.5}$
$\Rightarrow \lim_{x \to \infty} e(\epsilon) = -\infty$ (unbounded)

d) As the small (and unavoidable) error we make when we implement the initial conditions already leads to an unbounded relative error in the solution, the equation cannot be solved with an appropriate accuracy on a computer.

3 Stability of Second Order Methods

a) Method of Heun for $f(x, y) = \lambda \cdot y$:

$$\begin{aligned} y_k &= y_{k-1} + \frac{h}{2}\left[\lambda y_{k-1} + \lambda(y_{k-1} + h\lambda y_{k-1})\right] = \\ &= y_{k-1}\left(1 + \lambda \cdot h + \frac{\lambda^2}{2} \cdot h^2\right) = \\ &= \ldots = y_0 \left(1 + \lambda \cdot h + \frac{\lambda^2}{2} \cdot h^2\right)^k \end{aligned}$$

Thus, $|y_k| \leq |y_0|$ for all $k \Leftrightarrow \left|1 + \lambda \cdot h + \frac{\lambda^2}{2} \cdot h^2\right| \leq 1$.

$\Leftrightarrow -1 \leq 1 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2 \leq 1$

part 1:

$1 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2 \leq 1$.

$\Leftrightarrow \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2 \leq 0$

roots of $\lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2$:

$h_1 = 0, \quad h_2 = \dfrac{2}{|\lambda|}$.

$\dfrac{1}{2}\lambda^2 > 0 \quad \Rightarrow$

$\lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2 \leq 0 \quad \Leftrightarrow \quad 0 \leq h \leq \dfrac{2}{|\lambda|}.$

part 2:

$-1 \leq 1 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2$

$\Leftrightarrow 0 \leq 2 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2$

roots of $2 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2$:

$h_{1/2} = \dfrac{-\lambda \pm \sqrt{\lambda^2 - 4\lambda^2}}{\lambda^2}.$

$\Rightarrow$ no real roots

$\dfrac{1}{2}\lambda^2 > 0 \quad \Rightarrow \quad 2 + \lambda \cdot h + \dfrac{\lambda^2}{2} \cdot h^2 > 0$ for all h.

$\Rightarrow$ stability restrictions for the method of Heun: $h \leq \dfrac{2}{|\lambda|}$.

b) Implicit analogon of the method of Heun: Adams-Moulton

$y_{k+1} = y_k + \dfrac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right].$