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Introduction to Scientific Computing

Midterm

1 Convergence of the Midpoint Rule

Consider the ordinary differential equation

$$y'(x) = f(x, y)$$

with initial conditions

$$y(0) = y_0.$$

We want to compute approximations of the solution $y(x)$ at the discrete points $x_i = i \cdot h, i = 1, 2, 3, \ldots$ with a stepsize $h > 0$ and with the help of the midpoint rule

$$y_{k+1} = y_{k-1} + 2 \cdot h \cdot f(x_k, y_k).$$

a) Show that the midpoint rule is second order consistent!

b) Is the midpoint rule asymptotically stable?

c) Is the midpoint rule convergent? If so, what is the order of convergence?

2 Two-Dimensional Systems of Linear First Order Ordinary Differential Equations

The functions $y(x)$ and $z(x)$ are determined by the two-dimensional system of linear ordinary differential equations

$$\begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} -3 & 14 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} \tag{1}$$

and the initial conditions

$$y(-0.1) = y_i, \quad y'(-0.1) = y'_i.$$

a) Compute the general solution of (1) and determine the type of the critical point $(0, 0)$! Sketch some trajectories of solutions for different intial conditions!

b) Compute the first component $y(x)$ of the solution for

I) $y_i = e, \quad y'_i = -10e$ and

II) $y_i = e + \epsilon, \quad y'_i = -10e.$

For case I), the solution shall be called y, for case II), the solution is denoted by y_ϵ.

c) Compute the relative error

$$e(\epsilon) := \frac{y(x) - y_\epsilon(x)}{y(x)}.$$

Is it bounded for $x \in [0; \infty[$?

d) Is the model equation (1) with initial conditions $y_i = e, y_i' = -10e$ suitable for a solution algorithm on a computer?

3 Stability of Second Order Methods

We know from the lecture that the method of Heun

$$y_{k+1} = y_k + \frac{h}{2} \cdot [f(x_k, y_k) + f(x_{k+1}, y_k + h \cdot f(x_k, y_k))]$$

is a second order convergent method to approximate the solution of the initial value problem

$$y'(x) = f(x, y), \quad y(0) = y_0.$$

a) Consider the ordinary differential equation

$$y'(x) = \lambda \cdot y$$

with $\lambda < 0$ and the analytical solution $y(x) = a \cdot e^{\lambda \cdot x}$. Under which condition for the stepsize h is the stability condition

$$|y_k| \leq |y_0|$$

fullfilled for all $k \in 1, 2, 3, \ldots$?

b) Can you suggest a similar second order method for which there are no stability restrictions? What's the disadvantage of this method?

4 Wave Equation

5 Heat Energy