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Sheet 12
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Scientific Computing

Finite Element Methods

Exercise 27: Convection-Diffusion Equations

Consider the convection-diffusion equation for temperature transport in a fluid which moves at constant velocity $v \in \mathbb{R}$:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $D \in \mathbb{R}^+$ denotes the diffusion constant of the fluid. The problem shall be solved on the unit interval with homogeneous Dirichlet conditions.

(a) Derive the weak formulation of the equation. Discretise space by piecewise linear hat functions. Derive the semi-discrete set of equations and compute all coefficients. How can we categorise this set of equations?

(b) Perform mass-lumping to facilitate the time discretisation. Therefore, approximate the mass matrix $M_{ij} := \int \varphi_i \varphi_j dx$ by a diagonal matrix $\tilde{M}_{ij}$,

$$\tilde{M}_{ij} := \begin{cases} \sum\limits_j M_{ij} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Use the explicit Euler method to subsequently discretise the problem in time.

(c) Solve the problem with maple for $D = 1.0$, $v = 1.0$, a mesh size $h = 1/10$ and a time step $\tau = 0.002, 0.02$. Use initial conditions $T = 1$ inside the domain (and homogeneous Dirichlet conditions at the boundaries).

Exercise 28: Reference Elements

In the following, we want to compute the mapping from an arbitrary triangle E onto a reference triangle E_{ref}, cf. Fig. 1. This can be useful in several contexts, for example to simplify the integration procedures in the FE method or to prove error estimates for the respective finite elements and their basis functions.

(a) Define a transformation $\chi(\xi)$ which maps the coordinates ξ within the reference triangle E_{ref} (triangle on the right in Fig. 1) onto the triangle E (triangle on the left in Fig. 1). Use arbitrary coordinates P_0, P_1, $P_2 \in \mathbb{R}^2$ to define the transformation.

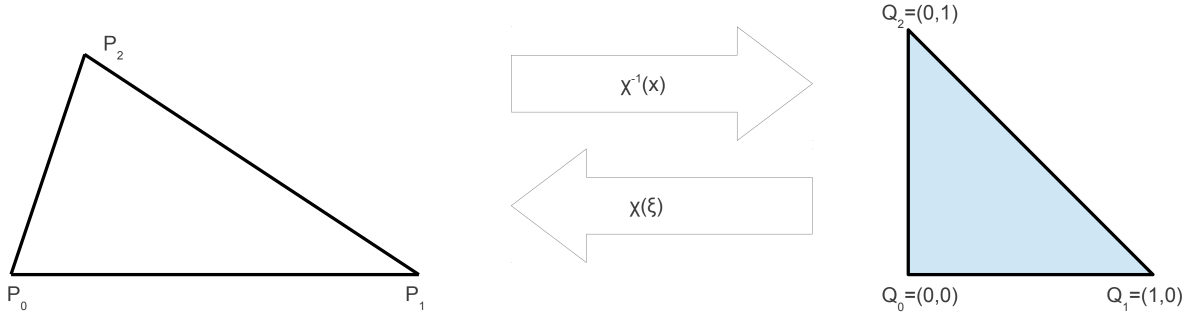

Figure 1: Coordinate mapping between a triangle of arbitrary shape (left) and a reference triangle (right).

(b) Denote the corners of the reference triangle by $Q_0 = (0,0)^\top$, $Q_1 = (1,0)^\top$, $Q_2 = (0,1)^\top$. Define linear functions $\Phi_i(\xi)$, $i = 0,1,2$, on the reference triangle such that $\Phi_i(Q_j) = \delta_{ij}$, that is

$$\Phi_i(Q_j) := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{3}$$

Compute the mass matrix $A^{\text{ref}}_{ij} := \int_{E_{\text{ref}}} \Phi_i(\xi)\Phi_j(\xi)d\xi$ of the reference element; you may use maple for this purpose.

(c) Use the u-substitution to derive a formula which evaluates the mass matrix $A_{ij} = \int_E \phi_i(x)\phi_j(x)dx$ for an arbitrary triangle and its respective basis functions $\phi_i(x)$. The formula may only make use of the transformation $\chi(\xi)$ and the mass matrix A^{ref} of the reference triangle.

(d) Validate your formula from task (c) by computing the mass matrix of the reference element from Exercise 26. You may use maple for this purpose.