Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel
M.Sc. D. Jarema

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Scientific Computing

Eigenvalues

Repetition

The eigenvalue theory can be used to characterise (amongst others) linear systems with respect to the amplification, reduction and frequency of the underlying matrix-based operations. The eigenvalues λ_i together with the corresponding eigenvectors v_i for a matrix $A \in \mathbb{R}^{N\times N}$ are all pairs for which hold: $A \cdot v_i = \lambda_i v_i$. It can be shown that the eigenvalues are identical with the roots of the characteristic polynomial $\det(A - \lambda \cdot \mathbb{1})$ where $\mathbb{1}$ denotes the eye-matrix; 'det' represents the *determinant* of the matrix $A - \lambda \cdot \mathbb{1}$. Hence, if λ_i is an eigenvalue, then it holds that $\det(A - \lambda_i \cdot \mathbb{1}) = 0$. Similar to the roots of other polynomials, the eigenvalues λ_i do not need to necessarily be real values; they might also lie in the space of complex numbers.
Depending on the properties of the matrix A, one can find out information about its eigenvalues. In the following, three examples should be given for such properties:

1 A matrix $A \in \mathbb{R}^{N\times N}$ is called *diagonalisable* if it can be written as $A = PDP^{-1}$ with invertible matrix $P \in \mathbb{R}^{N\times N}$ and diagonal matrix $D \in \text{diag}(N)$. In this case, the diagonal matrix contains the eigenvalues λ_i on its diagonal and the columns of P represent the corresponding eigenvectors.

2 A matrix $A \in \mathbb{R}^{N\times N}$ is called symmetric if it holds $A_{ij} = A_{ji}$ for all $i, j = 1, ..., N$. If a matrix is symmetric, all eigenvalues are real values.

3 A matrix $A \in \mathbb{R}^{N\times N}$ is called *positive definite* if $x^\top Ax > 0$ for all vectors $x \in \mathbb{R}^N \backslash \{\vec{0}\}$ and *positive semi-definite* if $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^N \backslash \{\vec{0}\}$. Analogous definitions hold for *negative (semi-)definiteness*. It holds the equivalence:

$$\text{matrix } A \text{ is positive definite} \Leftrightarrow \text{all eigenvalues are positive}$$

In the following exercises, we will practise the computation of eigenvalues and the characterisation of matrix-based systems in terms of their eigenvalues.

Remarks & Hints

- You may use the following rule to compute the determinant of a 2×2 matrix:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21} \tag{1}$$

- You may use the following rule to compute the determinant of a 3×3 matrix:

$$\begin{aligned} \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned} \tag{2}$$

Exercise 1: Direct Computation of Eigenvalues

Consider the matrices

$$A := \begin{bmatrix} -6 & -14 & -12 \\ 4 & 9 & 6 \\ 1 & 2 & 3 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

(a) Compute the eigenvalues of the matrices.

(b) Compute the eigenvectors of B.

(c) Which properties does the matrix B have?

Solution:
(a) Using the characteristic polynomial and the formula for the determinant from above yields for matrix A:

$$\begin{aligned} \det(A - \lambda \mathbb{1}) &= \det\left(\begin{bmatrix} -6-\lambda & -14 & -12 \\ 4 & 9-\lambda & 6 \\ 1 & 2 & 3-\lambda \end{bmatrix}\right) \\ &= (-6-\lambda)(9-\lambda)(3-\lambda) + (-14)\cdot 6 \cdot 1 + (-12)\cdot 4 \cdot 2 \\ &- 1\cdot(9-\lambda)(-12) - 2\cdot 6\cdot(-6-\lambda) - (3-\lambda)\cdot 4\cdot(-14) \\ &= ... = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \end{aligned} \tag{4}$$

An educated guess shows that the values $\lambda_0 = 1$, $\lambda_1 = 2$, $\lambda_2 = 3$ are the roots of the polynomial.
Analogously, one obtains the eigenvalues for the matrix B: $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$. Remark: The eigenvalues λ_0, λ_1 are trivial: it holds $B \cdot e_0 = e_0$ and $Be_2 = e_2$ for the Euclidean vectors $e_i = (0, ...0, 1, 0, ..., 0)^\top$.
(b) The eigenvectors v_i of B can be computed from the equation system $B \cdot v_i = \lambda_i v_i, i = 0, 1, 2$. We obtain $v_{0,1} = \{(w_0, w_1, w_2)^\top \in \mathbb{R}^3 : w_1 = w_0 + w_2\}$, $v_2 = \{(w_0, w_1, w_2)^\top \in \mathbb{R}^3 : w_0 = w_2 = 0\}$.
(c) B is not symmetric; though its eigenvalues are all bigger than zero, we can hence not immediately say if it's positive definite (following point 3 from the introduction on eigenvalues).

Population Models

Exercise 2: Rates and Calculation of Interest

Congratulations! You are just about to open a new bank account. To open it, you initially invest $K(n=0)$ euros, that is you start with $K(n=0)$ euros in the year $n=0$. After each year, you first obtain an interest rate of p % onto your current savings. Besides, you are obliged to add another J euros each year onto your current account.

(a) Try to find a model for your bank account which–based on a recursive formula–computes your savings $K(n+1)$ in the $(n+1)$-th year from the savings $K(n)$.

(b) Which value can be considered to be an eigenvalue in our recursive expression? Which quantities affect the eigenvalue and what happens to your savings when you modify them?

(c) Convert the recursive relation from (a) into a non-recursive expression.

(d) The people from the bank cheated on you. Though they first announced that the bank account is for free, you suddenly need to pay n euros in the year n, starting in the very first year (hence, only the first year was free)! Include the arising costs into the recursive expression from (a).

(e) How do the costs of n euros in year n enter the non-recursive formula for your savings?

(f) What can you buy from your saved money in 10 years, assuming an interest rate $p = 0.05$, an initial payment of $K(n=0) = 50$ euros and annual investments of $J = 50$? A new notebook ($\sim$ 1000 Euros), the latest iPad ($\sim$ 800 Euros), or the latest iPhone ($\sim$ 650 Euros)?

Solution

(a) $K(n+1) = (1+p) \cdot K(n) + J$

(b) The value $1+p$ can be considered to be a characteristic value and indeed is the eigenvalue of our recursive scheme for $J=0$. In this case, if $p<0$, it implies that we have to pay $-p$ of our savings to the bank in one year. For $n \to \infty$, our savings tend to zero, $K(n \to \infty) \to 0$. For $p>0$, we obtain more and more money, i.e. $K(n \to \infty) \to \infty$. For $p=0$, our savings stay constant: $K(n \to \infty) = K(0)$.

(c) We can first of all look at the first three years of our savings development:

$$\begin{aligned} K(1) &= (1+p)K(0) + J \\ K(2) &= (1+p)^2K(0) + (1+p)J + J \\ K(3) &= (1+p)^3K(0) + (1+p)^2J + (1+p)J + J \end{aligned} \tag{5}$$

We can hence write our problem as:

$$K(n+1) = (1+p)^{n+1}K(0) + J\sum_{k=0}^{n}(1+p)^k \tag{6}$$

The equality $\sum_{k=0}^{n} q^n = \frac{q^{n+1}-1}{q-1}$–applied to the last term of Eq. (6) yields:

$$K(n+1) = (1+p)^{n+1}K(0) + \frac{J}{p}\left((1+p)^{n+1} - 1\right) \tag{7}$$

(d) The new formula looks as follows: $K(n+1) = (1+p)K(n) + J - n$
(e) For the non-recursive formula, we can again consider the first three steps

$$\begin{aligned} K(1) &= (1+p)K(0) + J - 0 \\ K(2) &= (1+p)^2K(0) + (1+p)J + J - 0\cdot(1+p) - 1 \\ K(3) &= (1+p)^3K(0) + (1+p)^2J + (1+p)J + J - 0\cdot(1+p)^2 - 1\cdot(1+p) - 2, \end{aligned} \tag{8}$$

and–via induction–we obtain the formula

$$K(n+1) = (1+p)^{n+1}K(0) + J\sum_{k=0}^{n}(1+p)^k - \sum_{k=0}^{n} k(1+p)^{n-k} \tag{9}$$

The equality $\sum_{k=0}^{n} k\cdot q^k = \frac{n\, q^{n+2}-(n+1)q^{n+1}+q}{(q-1)^2}$ can be used to re-write the term

$$\sum_{k=0}^{n} k(1+p)^{n-k} = (1+p)^n \sum_{k=0}^{n} k\left(\frac{1}{1+p}\right)^k \tag{10}$$

with $q = 1/(1+p)$. Re-writing the term finally yields:

$$K(n+1) = (1+p)^{n+1}K(0) + \frac{J}{p}\left((1+p)^{n+1} - 1\right) - \frac{n-(n+1)(1+p)+(1+p)^{n+1}}{p^2} \tag{11}$$

(f) Inserting $p = 0.05$, $J = 50$ and $K(0) = 50$ into the formula from above yields: $K(10) = 658.78$. Hence, you can enjoy a new iPhone :-)

Exercise 3: Rabbits and Fibonacci Numbers

Let's consider the Fibonacci model for the evolution of rabbits. Each pair is assumed to have one pair of children each year (one male and one female rabbit). In their first year, the young rabbits do not have children. After the first year, they will also give birth to one pair of rabbits each year.

(a) Let Y denote the "young" rabbits and G the "grown-up" rabbits. Model the evolution of the rabbits by a recursive scheme and a respective linear relationship between the rabbits $Y(n), Y(n+1), G(n), G(n+1)$ of subsequent years $n, n+1$:

$$\begin{pmatrix} Y(n+1) \\ G(n+1) \end{pmatrix} = A \cdot \begin{pmatrix} Y(n) \\ G(n) \end{pmatrix} \tag{12}$$

with matrix $A \in \mathbb{R}^{2\times 2}$.

(b) Which properties does A have? Compute the eigenvalues and eigenvectors of A!

(c) Assume an initial rabbit population $(Y(0), G(0))^\top := (0,1)^\top$. How can we describe the evolution of the rabbits in terms of eigenvectors? How can we easily estimate the population in the year 20 by only considering the eigenvalues and eigenvectors of the system?
Hint: decompose the initial population $(Y(0), G(0))^\top$ into its eigenvector contributions and compute their evolution separately.

(d) Assume that each year p % of the grown-up rabbits and q % of the young rabbits die due to a big fat wolf in their forest. How can you include this assumption into the recursive model from above? How do the eigenvalues of the update matrix A change in this case?
Remark: for the sake of simplicity, assume that dying and giving birth is a strictly sequential process ;-), i.e. the respective percentage of rabbits dies first and the remaining rabbits give birth to new pairs of rabbits.

Solution
(a) Each new year, there will be as many young rabbits as grown-up rabbits: $Y(n+1) = G(n)$. Besides, the number of grown-up rabbits is arising from the original grown-ups $G(n)$ and the young rabbits of that year $Y(n)$, $G(n+1) = Y(n) + G(n)$. We hence obtain:

$$\begin{pmatrix} Y(n+1) \\ G(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} Y(n) \\ G(n) \end{pmatrix} \tag{13}$$

(b) The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is symmetric. The eigenvalues must hence be real-valued. The characteristic polynomial is given by $\det(A - \lambda \cdot \mathbb{1}) = \lambda^2 - \lambda + 1$. Its roots, that is the eigenvalues, are $\lambda_{0,1} = \frac{1\pm\sqrt{5}}{2}$.
(c) First, let's try to understand the hint from this exercise: assume we can write our initial condition $(Y(0), G(0))^\top$ as a linear combination of the two eigenvectors v_0, v_1 (that belong to λ_0 and λ_1, respectively), that is $(Y(0), G(0))^\top = c_0 \cdot v_0 + c_1 \cdot v_1$ with constants $c_0, c_1 \in \mathbb{R}$.

Then, we can compute the population in the year n via:

$$\begin{aligned}\begin{pmatrix} Y(n) \\ G(n) \end{pmatrix} &= A^n \begin{pmatrix} Y(0) \\ G(0) \end{pmatrix} \\ &= A^n(c_0 v_0 + c_1 v_1) \\ &= c_0 A^n v_0 + c_1 A^n v_1 \\ &= c_0 \lambda_0^n v_0 + c_1 \lambda_1^n v_1 \end{aligned} \tag{14}$$

From the last equation, it can be seen that we can track the evolution of the rabbits by considering the evolution of each eigenvector: depending on its corresponding eigenvalue, we can just compute λ_i^n and see how the eigenvector is either amplified or reduced over time. In our case, the eigenvalues are $\lambda_0 = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\lambda_1 = \frac{1-\sqrt{5}}{2} \approx -0.618$. For $n \to \infty$, the first term of the upper equation hence tends to infinity, $c_0 \lambda_0^n v_0 \overset{n\to\infty}{\to} \infty$, since the magnitude of the eigenvalue λ_0 is bigger than 1. For the second eigenvalue λ_1, it holds that $\|\lambda_1\| < 1$. The respective contribution $c_1 \lambda_1^n v_1$ is consequently decreasing over time, $c_1 \lambda_1^n v_1 \overset{n\to\infty}{\to} 0$.
In our example, the decomposition of the initial vector can be determined as $(0,1)^\top = c_0 \cdot v_0 + c_1 \cdot v_1$ with $c_0 = \frac{1}{\sqrt{5}}$, $c_1 = -\frac{1}{\sqrt{5}}$, $v_0 = (1, \frac{1+\sqrt{5}}{2})^\top$, $v_1 = (1, \frac{1-\sqrt{5}}{2})^\top$. For $n = 20$, the value λ_1^{20} is already so small that we can completely neglect the contribution of the respective summand to the overall population ($\lambda_1^{20} < 1e-4$). We obtain:

$$\begin{aligned}(Y(20), G(20))^\top &\approx c_0 \lambda_0^{20} v_0 \\ &\approx 0.45 \cdot 15100 \cdot (1, 1.62)^\top \\ &\approx (6800, 11000)^\top \end{aligned} \tag{15}$$

The "exact" iteration using the matrix A delivers: $(Y(20), G(20))^\top = A^{20}(Y(0), G(0))^\top = (6765, 10946)^\top$.
(d) As p% of the grown-up rabbits die, we will only have $1 - p/100$ new young rabbits, $Y(n+1) = (1 - p/100)G(N+1)$. With q % of the former young rabbits dying, we will have less grown-ups as well: $G(n+1) = (1 - q/100)Y(n) + (1 - p/100)G(n)$.
The arising matrix looks as follows:

$$A = \begin{pmatrix} 0 & 1 - \frac{p}{100} \\ 1 - \frac{q}{100} & 1 - \frac{p}{100} \end{pmatrix} \tag{16}$$

The eigenvalues evolve at $\lambda_{0,1} = \dfrac{1 - \frac{p}{100} \pm \sqrt{\left(1 - \frac{p}{100}\right)^2 - 4\left(-1 + \frac{p+q}{100} - \frac{pq}{10000}\right)}}{2}$.