

Scientific Computing

Eigenvalue Problems and Algebraic Models

Exercise 4: Economics (Model by Leontief)

In our city, we have n companies. Each company sells its own product x_j , $j = 1, \dots, n$. Besides, in order to produce one unit of its product, the company j requires A_{ij} units of the product from company i . The matrix A_{ij} , $i, j = 1, \dots, n$ hence describes the production relations within our city.

- (a) **Closed economic system:** Assume that our city is not involved in any trades with companies from other cities (no export or import). Each company hence uses all its incomes for further productions. How can we estimate the amount of products x_i which are required to have a stable economic system (that is everything that is produced is used for further productions again)?
- (b) Consider the production matrix:

$$A := \begin{pmatrix} \frac{3}{4} & \frac{1}{8} & \frac{19}{8} \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 2 \end{pmatrix} \quad (1)$$

Compute its eigenvalues and eigenvectors. Can the respective city–based on this production matrix–potentially survive as a closed economic system?

- (c) Since a new company will enter the city market in two years, the companies from (b) have to provide $x = (1.5, 1, 0)^\top$ products until then. Assume that the production matrix denotes the requirements that need to be fulfilled within one year. How many products do the companies need to provide at the present stage according to the Leontief model? How does this behavior relate to the eigenvalues of our system?
- (d) **Import and export:** Our city is now allowed to buy and sell products from/to other companies from outside. How can we modify our model to include this property? For which vector x do we then obtain a stable solution for the economic state of our city?

Solution:

- (a) Assume we know the amount of products x_j of each company. Per unit of product j , the

company requires A_{ij} units of the product i . Consequently, the company j needs to obtain $A_{ij}x_j$ from i . The needs for each company can hence be formulated as simple matrix-vector product, $b = Ax$, where b_i represents the need of company i (formed by the requirements of the companies $j = 1, \dots, n$).

In the closed economic system, we therefore obtain a stable economic state if all companies exactly produce enough to feed all other companies' and its own needs:

$$Ax = x \quad (2)$$

Hence, we look for the eigenvector of the eigenvalue $\lambda = 1$. If $(A - \mathbb{1})^{-1}$ exists, then we only have the zero vector as a solution to the problem above. Otherwise, whole subspaces may represent possible solutions to the problem, cf. exercise 4(b).

(b) The eigenvalues are computed as described on worksheet 1. The arising eigenvalues are:

$$\lambda_0 = 0.5, \lambda_1 = 1, \lambda_2 = 2$$

The respective eigenvectors are:

$$v_0 = (1, -2, 0)^\top, v_1 = (1, 2, 0)^\top, v_2 = (2, 1, 1)^\top$$

As there is an eigenvalue $\lambda_0 = 1$, we know that there exist stable solutions to the respective eigenvalue problem: if the three companies produce $x = c \cdot (1, 2, 0)^\top$ units of their products (only $c > 0$ makes sense in this context), the closed economic system will always remain in the current state.

(c) As the companies want to have x products in two years, they need to provide $y_{-1} = Ax$ products the year before and hence $y_{-2} = Ay_{-1} = A^2x$ products now.

The production vector x can be decomposed into the eigenvectors, $x = 0.5v_0 + v_1$. The solution can hence be obtained from:

$$y_{-2} = A^2(0.5v_0 + v_1) = 0.5\lambda_0^2v_0 + \lambda_1^2v_1$$

As $\lambda_0 < 1$, the first summand will become smaller over time. Assuming that the companies initially are (more or less) in the stable economic state, this implies that the adoption to the new state, i.e. to the state with the new company, becomes easier the more time they have. In this case, the contribution from v_1 is held constant over time whereas the disturbance v_0 can be driven towards the respective end state, depending on λ_0 and the number of years that the companies still have for this purpose.

(d) We add another vector which indicates the needs of external companies y_i from company i . To obtain a stable solution, we then need to have $x = Ax + y$. Instead of checking for the eigenvalue $\lambda = 1$, we need to search for a solution of $(A - \mathbb{1})x = -y$. If $(A - \mathbb{1})^{-1}$ exists, we have a unique solution.

Remark: Computing the inverse of a matrix can become very expensive. Approximations or iterative solution strategies hence are often preferable, especially in cases where the system is too big to be solved directly or when an approximate solution is sufficient. In case that all eigenvalues fulfill $0 < \lambda < 1$, the inverse of $A - \mathbb{1}$ can be approximated via $(A - \mathbb{1})^{-1} \approx \mathbb{1} + A + A^2 + \dots + A^n$.

Exercise 5: Sport Rankings

Alice and Bob like watching baseball games. They watch approx. one match per day. Since they are very big fans, they know all about the statistics of the teams (i.e. how often did team i win against j and vice versa; there are no draws).

- (a) Bob likes those teams who win. If his current favorite team wins, he will also watch this team's next match on the following day. If the team loses, Bob changes his mind and watches the match of the respective winner team. How often will Bob stay with one team in future?
- (b) With a probability β , a match needs to be canceled due to bad weather. Similar to Bob, Alice also watches the matches of the winner teams (hence, if her favorite team wins, she normally stays with this team as well and otherwise switches to the new winner team). If there is bad weather, Alice randomly chooses the team and will watch this team's match the next day. How long will Alice stay with each team?

Solution:

(a) We can model Bob's behavior by a Markov process. Let's start at a given configuration, i.e. Bob currently stays with team i . Let n denote the total number of teams. The probability to play against the team j is hence $\frac{1}{n-1}$. Bob knows from statistics that team i won g_{ij} times against team j and lost l_{ij} times. Bob will change to another team $j, j \neq i$, if team i plays against team j and loses. This happens with a probability

$$B_{ij} = \frac{1}{n-1} \cdot \frac{l_{ij}}{l_{ij} + g_{ij}}.$$

Bob stays with the team i if it wins:

$$B_{ii} = \sum_{i \neq j} \frac{1}{n-1} \cdot \frac{g_{ij}}{g_{ij} + l_{ij}}$$

The matrix B is a stochastic matrix. Starting from the given configuration, we can carry out a vector iteration $x(m) = Bx(m-1)$ with $x(0) = e_i$ (i -th Euclidean unit vector \Leftrightarrow Bob initially watches the match of team i). In the long term limit, we will obtain average values for Bob's behavior.

(b) Bob's model can be extended to Alice's case as well. Therefore, we can consider a "branching" of the probabilities. With a probability of β , there's rain and a new team is to be chosen randomly. Since the probability for each team is the same we have a probability of $\beta \cdot \frac{1}{n}$ for each team in this case. For the other case, we have the same concept as for Bob; still, we need to weight the probability by $1 - \beta$ to account for the fact that there's no rain. The final formula evolves at:

$$x(m) = (1 - \beta)Bx(m-1) + \beta \frac{1}{n}e$$

where $e = (1, 1, \dots, 1)^T$.