

Scientific Computing

Continuous Models: Ordinary Differential Equations

Exercise 9: Direction Fields

Consider the ordinary differential equation

$$\frac{dy(t)}{dt} = \lambda y(t)^2 + \mu y(t) - \nu$$

with real constants $\lambda, \mu, \nu \geq 0$.

- For $\lambda = 1, \mu = 0, \nu = 1$, compute the critical points, compute their characteristics (stable, unstable, saddle point) and sketch the respective direction field for $t \in [0, 4]$, $y \in [-2, 2]$.
- Write a maple sheet which sketches the direction fields of the ODE from above for arbitrary choices of λ, μ, ν .
- Compute the critical points of the ODE and characterise them using exemplary direction field plots of the maple sheet, i.e. for each relevant parameter combination, choose at least one parameter set, visualise the underlying direction field and determine the characteristics of the critical points.
- Consider the equation from above for the time-dependent parameter set $\lambda = 1, \mu = 4t$ and $\nu = 5t^2$ for $t > 0$. Modify your maple sheet for this scenario. What can you say about the critical points in this case?

Exercise 10: Exponential Function for Matrices

Similar to scalars, the exponential function can be extended to matrices. It is defined for a matrix $A \in \mathbb{R}^{N \times N}$ as

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

and can be used to analytically solve systems of ordinary differential equations.

Consider the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

- (a) Compute A^2, A^3, A^4 and A^5 and derive a general formula for A^{2k} and $A^{2k+1}, k \in \mathbb{N}$.
- (b) Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}^{2 \times 2}, f(t) := \exp(At)$. Show that

$$f(t) = I \cdot \cos(t) + A \cdot \sin(t)$$

where I is the identity matrix.

Hint: Use the results from (a) and consider the series representation of the trigonometric functions.