

Scientific Computing

Continuous Models: Ordinary Differential Equations

Exercise 9: Direction Fields

Consider the ordinary differential equation

$$\frac{dy(t)}{dt} = \lambda y(t)^2 + \mu y(t) - \nu$$

with real constants $\lambda, \mu, \nu \geq 0$.

- For $\lambda = 1, \mu = 0, \nu = 1$, compute the critical points, compute their characteristics (stable, unstable, saddle point) and sketch the respective direction field for $t \in [0, 4]$, $y \in [-2, 2]$.
- Write a maple sheet which sketches the direction fields of the ODE from above for arbitrary choices of λ, μ, ν .
- Compute the critical points of the ODE and characterise them using exemplary direction field plots of the maple sheet, i.e. for each relevant parameter combination, choose at least one parameter set, visualise the underlying direction field and determine the characteristics of the critical points.
- Consider the equation from above for the time-dependent parameter set $\lambda = 1, \mu = 4t$ and $\nu = 5t^2$ for $t > 0$. Modify your maple sheet for this scenario. What can you say about the critical points in this case?

Solution:

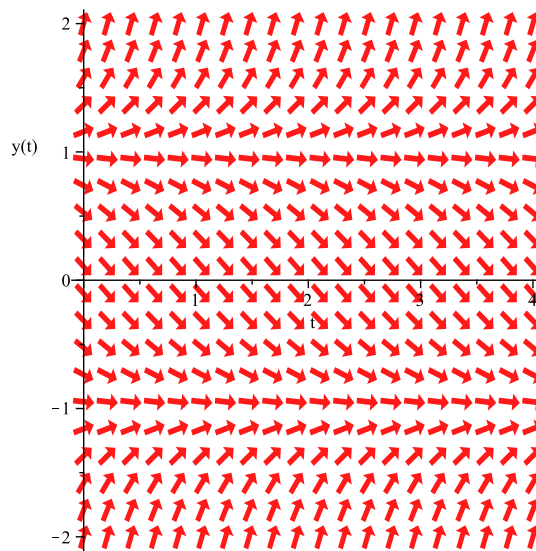
- The critical points evolve from $dy(t)/dt = y(t)^2 - 1 \stackrel{!}{=} 0 \Leftrightarrow y_{1,2}(t) = \pm 1$. To analyse the behaviour of the system, we take a small $\epsilon > 0$ and consider the deviation from the critical points:
 - For $y_1(t) = 1$:
 - For $y(t) := y_1(t) + \epsilon$: $dy/dt = (y_1(t) + \epsilon)^2 - 1 > 0$
 - For $y(t) := y_1(t) - \epsilon$: $dy/dt = (y_1(t) - \epsilon)^2 - 1 < 0$ (for sufficiently small ϵ)

According to the definition from the lecture, we see that $y_1(t)$ must be an unstable equilibrium point.

- For $y_2(t) = -1$:
 - For $y(t) := y_2(t) + \epsilon$: $dy/dt = (y_2(t) + \epsilon)^2 - 1 < 0$ (for sufficiently small ϵ)
 - For $y(t) := y_2(t) - \epsilon$: $dy/dt = (y_2(t) - \epsilon)^2 - 1 > 0$

According to the definition from the lecture, we see that $y_2(t)$ must be an attractive equilibrium point.

The sketch of the direction field looks as follows:



(b) See ws4_b.m

(c) The critical points evolve for $dy(t)/dt \stackrel{!}{=} 0$. First investigate the case that $\lambda \neq 0$:

$$\begin{aligned} \frac{dy(t)}{dt} &= \lambda y(t)^2 + \mu y(t) - \nu \stackrel{!}{=} 0 \\ \Leftrightarrow y_{1,2}(t) &= \frac{-\mu \pm \sqrt{\mu^2 + 4\lambda\nu}}{2\lambda} \end{aligned}$$

Hence, for $\mu = 0$ and $\nu = 0$, we have only one critical point at $-\mu/(2\lambda) \stackrel{\mu=0}{=} 0$. If we consider for example the set $\lambda = 1, \mu = \nu = 0$ and plot the direction field over $t = 0..20, y = -1..1$, we can observe that the arrows in the lower part ($y < 0$) tend towards $y(t) = 0$ whereas the direction field points away from the critical point in the upper

part of the graph ($y > 0$). We conclude that the respective critical point is a saddle point. For all other configurations of μ, ν , we obtain two critical points of the system. Let's consider the case $\lambda = 5, \mu = 4, \nu = 1$. The arising critical points are given by $y_1(t) = -1$ and $y_2(t) = \frac{1}{5}$. Plotting the arising direction fields over $t = 0..5, y = 0..0.8$, we see that $y_2(t)$ is unstable. For $t = 0..1, y = -2..0$, one can observe that the critical point $y_1(t)$ is asymptotically stable (attractive).

Now, let's turn to the case $\lambda = 0$. If $\lambda = 0$, we only have a linear equation system to solve for $y(t)$ and obtain:

$$\Leftrightarrow \begin{array}{lcl} \frac{dy(t)}{dt} & = & 0 \cdot y(t)^2 + \mu y(t) - \nu \stackrel{!}{=} 0 \\ & & y(t) \qquad \qquad \qquad = \frac{\nu}{\mu} \end{array}$$

Here, we need to assume that $\mu \neq 0$. Using the maple sheet, we see that this case corresponds to a stable or unstable point (for example, consider the direction field plot for $\lambda = 0, \mu = \pm 1, \nu = 1, t = 0..20, y = -5..5$). Finally, we consider the case where $\lambda = 0$ and $\mu = 0$. If $\nu \neq 0$, the equality $dy/dt \stackrel{!}{=} 0$ can never be reached. If $\nu = 0$, the respective equality is always fulfilled for all solutions since the arising solution is the constant curve, $y(t) = y_0$ with $y_0 := y(t = 0)$.

- (d) See WS4.d.m. Analogously to the case of constant coefficients from (c), we obtain two critical points:

$$y_{1,2}(t) = \frac{-4t \pm \sqrt{16t^2 + 20t^2}}{2} = -2t \pm 3|t| \stackrel{t \geq 0}{=} \begin{cases} -5t \\ t \end{cases}$$

Again, we plot the direction fields over suitable intervals: for $t = 0.1..2, y = -8..0$, we observe that $y_1(t) = -5t$ is an attractive critical point. For $t = 0.1..2, y = 0..2$, one can see that $y_2(t) = t$ is unstable.

Exercise 10: Exponential Function for Matrices

Similar to scalars, the exponential function can be extended to matrices. It is defined for a matrix $A \in \mathbb{R}^{N \times N}$ as

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

and can be used to analytically solve systems of ordinary differential equations.

Consider the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

- (a) Compute A^2, A^3, A^4 and A^5 and derive a general formula for A^{2k} and $A^{2k+1}, k \in \mathbb{N}$.
 (b) Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}^{2 \times 2}, f(t) := \exp(At)$. Show that

$$f(t) = I \cdot \cos(t) + A \cdot \sin(t)$$

where I is the identity matrix.

Hint: Use the results from (a) and consider the series representation of the trigonometric functions.

Solution:

(a) Computing the respective matrix products delivers:

$$\begin{aligned} A^2 &= -I \\ A^3 &= -A \\ A^4 &= I \\ A^5 &= A \end{aligned}$$

We can—via induction—conclude:

$$\begin{aligned} A^{2k} &= (-1)^k \cdot I \\ A^{2k+1} &= (-1)^k \cdot A \end{aligned}$$

(b) The trigonometric functions can be written as series as follows (see any good analysis text book):

$$\begin{aligned} \sin(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \\ \cos(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} \end{aligned}$$

We start with writing the function $f(t)$ according to the exponential series representation; here, we split the sum into odd and even summands:

$$f(t) = \exp(A \cdot t) = \sum_{k=0}^{\infty} \frac{1}{k!} (A \cdot t)^k = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} \cdot t^{2k+1}$$

From (a), we can insert the expressions for A^{2k} and A^{2k+1} :

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} \cdot t^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot I \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot A \cdot t^{2k+1} \\ &= I \cdot \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot t^{2k} + A \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot t^{2k+1} \end{aligned}$$

As the matrices I and A are not part of the summands anymore, we immediately see that the remaining two sums exactly correspond to the sin- and cos-definitions from above:

$$\begin{aligned} f(t) &= I \cdot \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot t^{2k} + A \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot t^{2k+1} \\ &= I \cdot \cos(t) + A \cdot \sin(t) \end{aligned}$$