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Scientific Computing

Continuous Models: Ordinary Differential Equations

First-Order vs. Higher-Order ODEs

An ordinary differential equation of the form

$$f(t, y(t), dy/dt, d^2y/dt^2, ..., d^ny/dt^n) = 0 \tag{1}$$

is called ODE *of order* n, i.e. the index of the highest derivative corresponds to the order of the ODE. In order to facilitate the analysis of ODEs, every ODE of order $n > 1$ can be transformed into a system of ODEs of order $n = 1$ as follows:

1. Assume our ODE from Eq. (1) to have order $n > 1$. Then, we can introduce helper functions $y_0, ..., y_{n-1}$:

$$\begin{aligned} y_0(t) &:= y(t) \\ y_1(t) &:= \frac{dy(t)}{dt} \\ &\vdots \\ y_{n-1} &= \frac{d^{n-1}y(t)}{dt^{n-1}} \end{aligned}$$

2. Using the helper functions, we can transform the ODE from Eq. (1) into the following system of ODEs:

$$\begin{aligned} \frac{dy_0}{dt} &= y_1 \\ \frac{dy_1}{dt} &= y_2 \\ &\vdots \\ f(t, y_0, \frac{dy_0}{dt}, \frac{dy_1}{dt}, ..., \frac{dy_{n-1}}{dt}) &= 0 \end{aligned}$$

The latter system only consists of first-order derivatives with respect to the helper functions.

Exercise 11: Transformation of Higher-Order ODEs

Consider the ODE

$$\frac{d^2y}{dt^2} = -y \tag{2}$$

which should be defined on an interval $t \in [0, \pi/2]$.

(a) Determine the order of this ODE.

(b) Transform the ODE into a system of first-order ODEs. Write the transformed system as

$$\begin{pmatrix} \frac{dy_0}{dt} \\ \vdots \\ \frac{dy_{n-1}}{dt} \end{pmatrix} = A \cdot \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3}$$

with matrix $A \in \mathbb{R}^{n \times n}$.

(c) Similar to the one-dimensional case, *homogeneous systems of ODEs*, that is ODEs which are of the form from Eq. (3), can be solved analytically using the exponential function for matrices. In case of Eq. (3) and initial conditions $y_0(0) = c_0, y_1(0) = c_1, \dots, y_{n-1}(0) = c_{n-1}$, the solution is given by:

$$\begin{pmatrix} y_0(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = \exp(A \cdot t) \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{4}$$

Use the results from exercise 10 to determine the general solution of Eq. (2). Which information is required to obtain a unique solution? Sketch at least two approaches to obtain a unique solution for the given ODE!

Exercise 12: Critical Points and Direction Fields for Two-Dimensional ODE Systems

(a) Consider the ODE from Eq. (2) and its respective matrix-vector form (Eq. (3)). What can you say about its critical points and stability? Draw the direction field on $[-1;1] \times [-1;1]$.

(b) Write a maple sheet which plots the direction field to validate your theoretical results from (a).

(c) Consider the modified ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t)$$

with $\mu \geq 0$. How does the parameter μ affect the critical point, stability and the direction field from (a)? Modify your maple sheet for this purpose.

(d) Consider the ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t) + \frac{dy(t)}{dt}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. What happens to the critical point, stability and direction field in this case?