Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel
M.Sc. D. Jarema

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Scientific Computing

Continuous Models: Ordinary Differential Equations

First-Order vs. Higher-Order ODEs

An ordinary differential equation of the form

$$f(t, y(t), dy/dt, d^2y/dt^2, ..., d^ny/dt^n) = 0 \tag{1}$$

is called ODE *of order n*, i.e. the index of the highest derivative corresponds to the order of the ODE. In order to facilitate the analysis of ODEs, every ODE of order $n > 1$ can be transformed into a system of ODEs of order $n = 1$ as follows:

1. Assume our ODE from Eq. (1) to have order $n > 1$. Then, we can introduce helper functions $y_0, ..., y_{n-1}$:
$$\begin{aligned} y_0(t) &:= y(t) \\ y_1(t) &:= \frac{dy(t)}{dt} \\ &\vdots \\ y_{n-1} &= \frac{d^{n-1}y(t)}{dt^{n-1}} \end{aligned}$$

2. Using the helper functions, we can transform the ODE from Eq. (1) into the following system of ODEs:
$$\begin{aligned} \frac{dy_0}{dt} &= y_1 \\ \frac{dy_1}{dt} &= y_2 \\ &\vdots \\ f(t, y_0, \frac{dy_0}{dt}, \frac{dy_1}{dt}, ..., \frac{dy_{n-1}}{dt}) &= 0 \end{aligned}$$

The latter system only consists of first-order derivatives with respect to the helper functions.

Exercise 11: Transformation of Higher-Order ODEs

Consider the ODE

$$\frac{d^2y}{dt^2} = -y \tag{2}$$

which should be defined on an interval $t \in [0, \pi/2]$.

(a) Determine the order of this ODE.

(b) Transform the ODE into a system of first-order ODEs. Write the transformed system as

$$\begin{pmatrix} \frac{dy_0}{dt} \\ \vdots \\ \frac{dy_{n-1}}{dt} \end{pmatrix} = A \cdot \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3}$$

with matrix $A \in \mathbb{R}^{n \times n}$.

(c) Similar to the one-dimensional case, *homogeneous systems of ODEs*, that is ODEs which are of the form from Eq. (3), can be solved analytically using the exponential function for matrices. In case of Eq. (3) and initial conditions $y_0(0) = c_0, y_1(0) = c_1, \ldots, y_{n-1}(0) = c_{n-1}$, the solution is given by:

$$\begin{pmatrix} y_0(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = \exp(A \cdot t) \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{4}$$

Use the results from exercise 10 to determine the general solution of Eq. (2). Which information is required to obtain a unique solution? Sketch at least two approaches to obtain a unique solution for the given ODE!

Solution:

(a) The index of the highest derivative in Eq. (2) is two. Hence, the order of the ODE is $n = 2$.

(b) We introduce two functions $y_0(t)$ and $y_1(t)$ as follows:

$$\begin{aligned} y_0(t) &= y(t) \\ y_1(t) &= \frac{dy(t)}{dt} \end{aligned}$$

From this, we obtain the following relation for the first derivatives of y_0, y_1:

$$\begin{aligned} \frac{dy_0(t)}{dt} &= \frac{dy(t)}{dt} = & y_1(t) &= 0 \cdot y_0(t) + 1 \cdot y_1(t) \\ \frac{dy_1(t)}{dt} &= \frac{d^2y(t)}{dt^2} = & -y(t) = -y_0(t) &= -1 \cdot y_0(t) + 0 \cdot y_1(t) \end{aligned}$$

Now, we can write our first-order system of ODEs in the matrix-vector form:

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix}$$

with matrix

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

(c) From exercise 10, we know that the exponential $\exp(A \cdot t)$ for our matrix A from above has the form

$$\exp(A \cdot t) = I \cdot \cos(t) + A \cdot \sin(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

The general solution of our ODE hence looks as follows:

$$\begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

Different possibilities exist to determine a unique solution, for example:

- Fix the solution of $y(t) = y_0(t)$ *at both sides* of the defined interval, i.e. prescribe $y(0) := K_0$ and $y(\pi/2) := K_1$. Inserting these relations into the first equation of our system delivers $c_0 = K_0$, $c_1 = K_1$.
- Fix the solution of $y(t) = y_0(t)$ *at one side and also fix its derivative*. For this purpose, we can prescribe $y(0) := K_0$ and $dy(0)/dt = y_1(0) := K_1$. Inserting these relations into the solution of the ODE system from above yields $c_0 = K_0$ and $c_1 = K_1$.

Exercise 12: Critical Points and Direction Fields for Two-Dimensional ODE Systems

(a) Consider the ODE from Eq. (2) and its respective matrix-vector form (Eq. (3)). What can you say about its critical points and stability? Draw the direction field on $[-1;1] \times [-1;1]$.

(b) Write a maple sheet which plots the direction field to validate your theoretical results from (a).

(c) Consider the modified ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t)$$

with $\mu \geq 0$. How does the parameter μ affect the critical point, stability and the direction field from (a)? Modify your maple sheet for this purpose.

(d) Consider the ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t) + \frac{dy(t)}{dt}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. What happens to the critical point, stability and direction field in this case?

Solution:

(a) The eigenvalues of the underlying matrix

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

are given by $\lambda = i$. The eigenvalue is complex and its real part is zero. Following the table on "Stability of Linear Systems" (cf. Slide 27 of the lecture 04_population2.pdf), the system is therefore expected to be stable and to have a "centered" critical point. Since A has full rank and since our system is homogeneous (the latter corresponds to $b = \vec{0}$ on slide 25, 04_population2.pdf), the critical point is given by $A^{-1}\vec{0} = \vec{0}$.

(b) See ws_5b.mw.

(c) See ws_5c.mw. The eigenvalues arise in this case to be $\mu = \sqrt{\mu}i$. The stability is—according to the overview from the lecture—still guaranteed. The matrix A has full rank for all $\mu \neq 0$; the critical point remains at $\vec{0}$ in this case. The direction field has an ellipsoidal shape.
For $\mu = 0$, the ODE reads $d^2y(t)/dt^2 = 0$ and has the solution $y(t) = c \cdot t + d$ with constants $c, d \in \mathbb{R}$. The only eigenvalue of A is $\lambda = 0$ in this case.

(d) See ws_5d.mw. Re-writing the ODE as a system of first-order ODEs yields a homogeneous system $(y_0(t), y_1(t))^\top = A \cdot (y_0(t), y_1(t))^\top$ with matrix

$$A := \begin{pmatrix} 0 & 1 \\ -\mu & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_{0,1} = \dfrac{1 \pm \sqrt{1-4\mu}}{2}$. Since $\mu \neq 0$, the inverse of A exists and is given by

$$A^{-1} := \frac{1}{\mu} \begin{pmatrix} 1 & -1 \\ \mu & 0 \end{pmatrix}.$$

For

- $\mu = 0.25$, we have only one real eigenvalue $\lambda_0 = \lambda_1 = 0.5 > 0$. From the overview in the lecture, we expect the system to have a single node as critical point and unstable behaviour. Since the system is homogeneous, the critical point is again given by $\vec{0}$.
- $\mu < 0.25$, we have two real-valued eigenvalues. If
  - $0 < \mu < 0.25$, all eigenvalues are bigger than zero $\Rightarrow$ critical point is a node, system is unstable
  - $\mu < 0$, one eigenvalue is bigger and one eigenvalue is smaller than zero $\Rightarrow$ critical point is a saddle point, system is unstable
- $\mu > 0.25$, we have complex eigenvalues, $\lambda_{0,1} = \frac{1}{2} \pm \sqrt{4\mu - 1}i$. Since the real part of the eigenvalues is always bigger than zero, we expect a spiral point and unstable behaviour.