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Sheet 6
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Scientific Computing

Ordinary Differential Equations: Numerical Methods

Exercise 13: Convergence of the Euler Method

Consider the ODE

$$\frac{dy(t)}{dt} = Ay(t) + b \tag{1}$$

with $A \in \mathbb{R}^{N\times N}$, $y(t) : \mathbb{R}^N \to \mathbb{R}^N$ and $b \in \mathbb{R}^N$ (this could for example be the linear system arising from the two-species model). The explicit Euler method applied to this equation reads:

$$y^{(n+1)} = y^{(n)} + \tau(Ay^{(n)} + b) \tag{2}$$

with time step τ and $y^{(n)} := y(n \cdot \tau)$.

(a) Show the following statement: if the Euler method converges towards a vector y^*, then y^* must be a critical point of the ODE.

(b) Under which conditions does the Euler discretisation from above (Eq. (2)) converge towards a critical point y^*?

Exercise 14: Analysis of Single-Step Methods

Consider the ODE from last time

$$\frac{d^2y}{dt^2} = -y$$

and its transform into a first-order system of ODEs

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} \tag{3}$$

(a) Formulate the discrete update rule for the first-order system of Eq. (3) when applying the following single-step methods and using a time step τ:

- explicit Euler method
- implicit Euler method

- trapezoidal rule (Crank-Nicolson)

Write down the respective update scheme in matrix-vector form as

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = A_{method} \cdot \begin{pmatrix} y_0^{n} \\ y_1^{n} \end{pmatrix} \tag{4}$$

where A_{method} denotes the method- and time step-dependent matrix for each of the single-step methods from above and $y^n := y(n \cdot \tau)$. What can you say about the long-time behaviour of the system, that is for (y_0^n, y_1^n) when $n \to \infty$?

(b) Write a maple sheet and check your analytical findings. You may consider solving the ODE from Eq. (3) for the initial values $y(0) = 0, dy(0)/dt = 1$.