Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel
M.Sc. D. Jarema

WS 2013/14
Sheet 9
18.12.2013

Scientific Computing

Partial Differential Equations

Exercise 19: Von Neumann Stability Analysis

Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1)$$

together with boundary conditions $u(t,0) = u(t,1) = 0$. We apply two different discretisation schemes using either explicit or implicit Euler time-stepping and the standard second-order approximation of the spatial derivative:
Explicit:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = \frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{h^2} \quad (1)$$

Implicit:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = \frac{u_{j-1}^{(m+1)} - 2u_j^{(m+1)} + u_{j+1}^{(m+1)}}{h^2} \quad (2)$$

where h and τ denote meshsize and time step and $u_j^{(m)} := u(m\tau, jh)$.
According to the von Neumann stability analysis, we assume the error in the solution to be of the type

$$u_j^{(m)} = (a_k)^m \sin(\pi k(jh)).$$

(a) Derive an explicit formula for the coefficient a_k in case of the explicit time-stepping. You may use the equality $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$.

(b) Typically, the error is not given by a single frequency, but by a superposition of several frequencies,

$$u_j^{(m)} = \sum_k c_k (a_k)^m \sin(\pi kjh).$$

Why is it enough to only consider single frequencies? What is the maximum frequency that we need to consider?

(c) The error in the solution decays if it holds $|a_k| < 1$ for all coefficients a_k. Which condition do we have to satisfy for the explicit time-stepping scheme in order to achieve this?

(d) Carry out the analysis from (c) for the implicit time-stepping scheme. Which condition arises in this case?

Exercise 20: Wave Equation

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $x \in (0, 1)$ with initial conditions

$$u(t = 0, x) = e^{-100(x-0.4)^2}, \quad \frac{\partial u(t = 0, x)}{\partial t} = 0$$

and boundary conditions

$$u(t, x = 0) = u(t, x = 1) = 0.$$

Write a maple sheet to solve this problem using finite differences on an equidistant grid with $N + 1$ grid points. The meshsize is denoted by $h := 1/N$ and the time step by τ.

(a) Discretise the temporal and spatial derivatives and formulate an update rule

$$u((m + 1)\tau, ih) = f\left(u(m\tau, ih), u(m\tau, (i + 1)h), u(m\tau, (i - 1)h), u((m - 1)\tau, ih)\right)$$

where $i \in \{1, ..., N - 1\}$, $m \in \mathbb{N}$.

(b) How can we include the initial condition?

(c) A formula that incorporates this particular initial condition and also preserves the accuracy of the scheme is given by

$$u(\tau, ih) := u(0, ih) + \frac{\tau^2}{2h^2}(u(0, (i - 1)h) - 2u(0, ih) + u(0, (i + 1)h)$$

and setting $u(t = 0, ih)$ as described above. Implement both your and this approach for the initial conditions in a maple sheet and solve the problem for $\tau \in \{0.01, 0.02\}$, $N = 90$. Plot the solution in every time step and compare the results according to your initial condition. You may consider the time interval $t \in [0, 1]$. What do you observe?