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Scientific Computing

Partial Differential Equations

Exercise 19: Von Neumann Stability Analysis

Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,1)$$

together with boundary conditions $u(t,0) = u(t,1) = 0$. We apply two different discretisation schemes using either explicit or implicit Euler time-stepping and the standard second-order approximation of the spatial derivative:

Explicit:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = \frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{h^2} \tag{1}$$

Implicit:

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\tau} = \frac{u_{j-1}^{(m+1)} - 2u_j^{(m+1)} + u_{j+1}^{(m+1)}}{h^2} \tag{2}$$

where h and τ denote meshsize and time step and $u_j^{(m)} := u(m\tau, jh)$.

According to the von Neumann stability analysis, we assume the error in the solution to be of the type

$$u_j^{(m)} = (a_k)^m \sin(\pi k(jh)).$$

(a) Derive an explicit formula for the coefficient a_k in case of the explicit time-stepping. You may use the equality $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$.

(b) Typically, the error is not given by a single frequency, but by a superposition of several frequencies,

$$u_j^{(m)} = \sum_k c_k (a_k)^m \sin(\pi k j h).$$

Why is it enough to only consider single frequencies? What is the maximum frequency that we need to consider?

(c) The error in the solution decays if it holds $|a_k| < 1$ for all coefficients a_k. Which condition do we have to satisfy for the explicit time-stepping scheme in order to achieve this?

(d) Carry out the analysis from (c) for the implicit time-stepping scheme. Which condition arises in this case?

Solution:

(a) Inserting the definition of $u_j^{(m)}$ into the explicit time-stepping scheme yields:

$$\begin{aligned}
u_j^{(m+1)} &= a_k^{m+1}\sin(\pi kjh) \\
&\overset{!}{=} u_j^{(m)} + \tau\frac{u_{j-1}^{(m)} - 2u_j^{(m)} + u_{j+1}^{(m)}}{h^2} \\
&= a_k^m \sin(\pi kjh) + \\
&\quad \frac{\tau}{h^2}\left(a_k^m \sin(\pi k(j-1)h) - 2a_k^m \sin(\pi kjh) + a_k^m \sin(\pi k(j+1)h)\right) \\
&\overset{\substack{\sin(A+B)+\sin(A-B)\\ =2\sin(A)\cos(B)}}{=} a_k^m \sin(\pi kjh) + \frac{\tau}{h^2}\left(a_k^m \cdot 2 \cdot \sin(\pi kjh)\cos(\pi kh) - 2a_k^m \sin(\pi kjh)\right)
\end{aligned}$$

Dividing by $a_k^m \sin(\pi kjh)$ results in an expression for a_k:

$$a_k = 1 + \frac{2\tau}{h^2}(\cos(\pi kh) - 1)$$

(b) It is enough to consider a single frequency since both the continuous and the discrete representation of the heat equation are *linear equations*. If we have two functions u and v which satisfy Eq. (1), then $\lambda \cdot u + \mu \cdot v$ with coefficients λ, μ also satisfy the equation. Assume we have an equidistant discretisation in space with $N+1$ points; the meshsize is $h = 1/N$, respectively. For each point jh, consider the frequencies $\sin(\pi \cdot 0 \cdot (jh)) = 0$ and $\sin(\pi \cdot N(jh)) = \sin(\pi \cdot j) = 0$. Both functions are zero in all grid points which implies that the high frequency $\sin(\pi N x)$ cannot be sufficiently resolved on our grid.

(c) It is enough to investigate a_k for $0 < k < N$. If $k = 0$, the corresponding error frequency is already zero. If $k \geq N$, the frequency cannot be resolved on our grid. For all cases $0 < k < N$, we see that the term $\cos(\pi kh) \in (-1, 1)$ is monotonously decreasing on this interval for increasing k. We can hence check the behaviour of a_k at the respective boundaries $\cos(\pi kh) \overset{!}{=} \pm 1$.
For $\cos(\pi kh) = 1$, we obtain $a_k = 1$, and for $\cos(\pi kh) = -1$, we have $a_k = 1 - \frac{4\tau}{h^2} < 1$. Due to the monotonous decrease of a_k, we see that $a_k < 1$ is always fulfilled on the interval $\cos(\pi kh) \in (-1, 1)$. In order to satisfy $a_k > -1$, it is required that

$$1 - \frac{4\tau}{h^2} > -1 \quad \Leftrightarrow \frac{\tau}{h^2} < \frac{1}{2}.$$

If we want to reduce our spatial resolution by a factor of two, we need to reduce our time step by a factor of four to remain stable!

(d) For the implicit time-stepping scheme, we can again compute the coefficients a_k the same way as for the explicit scheme. The coefficients evolve at

$$a_k = \frac{1}{1 - \frac{2\tau}{h^2}(\cos(\pi kh) - 1)}.$$

Using the monotony of $\cos(\pi kh)$ as in the previous part of this exercise, it is enough to consider the extreme cases $\cos(\pi kh) \overset{!}{=} \pm 1$. We obtain

$$a_k \overset{\cos(\pi kh)\overset{!}{=}1}{=} 1$$

$$a_k \overset{\cos(\pi kh)\overset{!}{=}-1}{=} \frac{1}{1+\frac{4\tau}{h^2}}$$

On the interval $\cos(\pi kh) \in (-1, 1)$, $|a_k| < 1$ is consequently fulfilled for *all choices of τ and h*. Our time-implicit algorithm is hence unconditionally stable.

Exercise 20: Wave Equation

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $x \in (0, 1)$ with initial conditions

$$u(t = 0, x) = e^{-100(x-0.4)^2}, \quad \frac{\partial u(t = 0, x)}{\partial t} = 0$$

and boundary conditions

$$u(t, x = 0) = u(t, x = 1) = 0.$$

Write a maple sheet to solve this problem using finite differences on an equidistant grid with $N + 1$ grid points. The meshsize is denoted by $h := 1/N$ and the time step by τ.

(a) Discretise the temporal and spatial derivatives and formulate an update rule

$$u((m + 1)\tau, ih) = f\left(u(m\tau, ih), u(m\tau, (i + 1)h), u(m\tau, (i - 1)h), u((m - 1)\tau, ih)\right)$$

where $i \in \{1, ..., N - 1\}$, $m \in \mathbb{N}$.

(b) How can we include the initial condition?

(c) A formula that incorporates this particular initial condition and also preserves the accuracy of the scheme is given by

$$u(\tau, ih) := u(0, ih) + \frac{\tau^2}{2h^2}(u(0, (i - 1)h) - 2u(0, ih) + u(0, (i + 1)h)$$

and setting $u(t = 0, ih)$ as described above. Implement both your and this approach for the initial conditions in a maple sheet and solve the problem for $\tau \in \{0.01, 0.02\}$, $N = 90$. Plot the solution in every time step and compare the results according to your initial condition. You may consider the time interval $t \in [0, 1]$. What do you observe?

Solution:

(a) Standard discretisation using the second-order finite differences (we set $u_i^{(m)} := u(m\tau, ih)$)

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i-1}^{(m)} - 2u_i^{(m)} + u_{i+1}^{(m)}}{h^2}$$
$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_i^{(m+1)} - 2u_i^{(m)} + u_i^{(m-1)}}{\tau^2}$$

yields:

$$u_i^{(m+1)} = \frac{\tau^2}{h^2}\left(u_{i+1}^{(m)} + u_{i-1}^{(m)}\right) + 2\left(1 - \frac{\tau^2}{h^2}\right) u_i^{(m)} - u_i^{(m-1)}$$

(b) The problem we have to face is that, in order to start the time-stepping, we need to provide at least *two values of u over time*. Therefore, we need to incorporate the gradient expression $\frac{\partial u(t=0,x)}{\partial t} = 0$ into our initial condition. The simplest approach using finite differences is to approximate the gradient by

$$0 = \frac{\partial u(t=0,x)}{\partial t} \approx \frac{u_i^{(1)} - u_i^{(0)}}{\tau}.$$

We can hence set $u_i^{(0)}$ according to the given function from above and $u_i^{(1)} = u_i^{(0)}$.

(c) See ws9_20.mw. For $\tau = 0.02$, the solution becomes unstable and explodes. It can be shown by stability analysis that the time step and the meshsize need to satisfy $\tau/h \leq 1$ for the present case. For $\tau = 0.01$ and $N = 90$, we have $\tau/h = 0.9 < 1$ and for $\tau = 0.02$, we obtain $\tau/h = 1.8 > 1$; the latter is beyond the stability range. The difference in the solution at $t = 1$ for the two different initial conditions is not visible from the default maple plots. Still, it should be noted that the simple initial condition does not have same order of accuracy (O(τ) instead of O(τ^2) as in the second case).

Next, we will find the restrictions of the time step size from the von Neumann stability analysis. We constitute u_i^m by error $a_k^m \sin(\pi kh)$ in the numerical scheme

$$\begin{aligned}
a_k^{(m+1)} sin(\pi kih) &= \frac{\tau^2}{h^2}(a_k^m \sin(\pi k(i+1)h) + a_k^m \sin(\pi k(i-1)h)) \\
&+ 2\left(1 - \frac{\tau^2}{h^2}\right) a_k^m \sin(\pi kih) - a_k^{m-1} \sin(\pi kih) \\
&= \frac{2\tau^2}{h^2} a_k^m \sin(\pi kih)\cos(\pi kh) + 2\left(1 - \frac{\tau^2}{h^2}\right) a_k^m \sin(\pi kih) \\
&- a_k^{m-1} \sin(\pi kih).
\end{aligned}$$

After simplification we get a quadratic equation for a_k

$$a_k^2 - 2\left(\frac{\tau^2}{h^2}(\cos(\pi kh) - 1) + 1\right) a_k + 1 = 0.$$

The solution of this equation is

$$a_k = \left(1 - \frac{\tau^2}{h^2}(1 - \cos(\pi k h))\right) \pm \left[\left(1 - \frac{\tau^2}{h^2}(1 - \cos(\pi k h))\right)^2 - 1\right]^{\frac{1}{2}}.$$

By using the following notation $b = 1 - \frac{\tau^2}{h^2}(1 - \cos(\pi k h))$ we obtain

$$a_k = b \pm \left[b^2 - 1\right]^{\frac{1}{2}}.$$

If $|b| \leq 1$ then $a_k = b \pm \mathrm{i}\left[|1 - \mathrm{b}^2|\right]^{\frac{1}{2}}$ and the module of the error amplitude is $|a_k| = b^2 + 1 - b^2 = 1$. So the numerical scheme is neutrally stable.

When $|b| > 1$ we have to consider two cases: $b > 1$ and $b < -1$.

In the first case we always get $a_k = b + \left[b^2 - 1\right]^{\frac{1}{2}} > 1$. And in the second case $a_k = b - \left[b^2 - 1\right]^{\frac{1}{2}} < -1$. To summarize, the numerical scheme is unstable when $|b| > 1$.

We have just to analyse $|b| \leq 1$ or $\left|1 - \frac{\tau^2}{h^2}(1 - \cos(\pi k h))\right| \leq 1$. Due to the monotony of $\cos(\pi k h)$ it is enough to consider two extreme cases $\cos(\pi k h) \stackrel{!}{=} \pm 1$.

$$b \stackrel{\cos(\pi k h) \stackrel{!}{=} 1}{=} 1,$$

$$b \stackrel{\cos(\pi k h) \stackrel{!}{=} -1}{=} 1 - \frac{2\tau^2}{h^2}.$$

In the second case b is always less than one. Therefore, the restriction for the time step size we get from the inequality $1 - 2\tau^2/h^2 \geq -1$, which yields $\tau/h \leq 1$.