Scientific Computing I

Module 3: Population Modelling – Continuous Models (Parts I and II)

Tobias Neckel

Winter 2015/2016

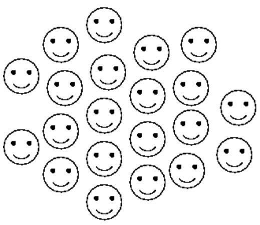
Part I: ODE Models

Part II: Discussion and Analysis of ODE Models

Part I

ODE Models

Discrete vs. Contiuous Models



$$\frac{dp}{dt} = F(p, t, \dots)$$

$$p(t) = ?$$

discrete model: $p(t) \in \mathbb{N}$ individuals

continuous model: $p : \mathbb{R} \to \mathbb{R}, p(t) = ?$

Move to Continuous Models:

- easier(?) type of mathematical problem: differential equations, calculus
- analytical solutions available(?)

Model of Malthus (1798)

Only one species:

1. birth rate γ (number of births per time interval) proportional to size of population
2. death rate δ proportional to size of population
3. thus: constant growth (or decay) rate: $r = \gamma - \delta$

Modelling:

- constant growth rate

$$\frac{\Delta p}{\Delta t} = r \cdot p \qquad \text{or} \qquad \frac{dp}{dt} = r \cdot p$$

- growth within a time interval (cmp. Taylor series)

$$p(t + \Delta t) = p(t) + \Delta p(t) = p(t) + r \cdot p(t) \cdot \Delta t$$

Model of Malthus – Differential Equation

- Move to infinitesimally small time steps – note: $dp = dp(dt)$:

$$\lim_{dt \to 0} \frac{dp}{dt} = r \cdot p$$

- Written as an ordinary differential equation:

$$\dot{p}(t) = r \cdot p(t)$$

- Requires initial condition (population at start):

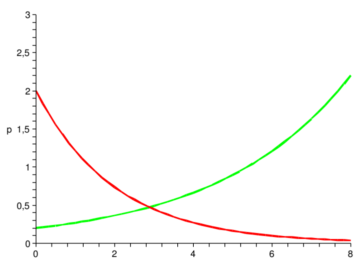
$$p(0) = p_0$$

- Analytical solution:

$$p(t) = p_0 \, e^{r\,t}$$

Model of Malthus – Solutions

The model of Malthus describes exponential growth or decay of a population:



Model of Verhulst (19th century)

Objective:

- model populations that approach saturation value

Assumptions:

- growth/death term depends on population size; assume linear dependency:

$$g(t) = g_0 - g_1 \cdot p(t) \qquad d(t) = d_0 + d_1 \cdot p(t)$$

- leads to differential equation:

$$\dot{p}(t) = g(t) - d(t) = \underbrace{(g_0 - d_0)}_{=:\alpha} - \underbrace{(g_1 + d_1)}_{=:\beta} p(t)$$

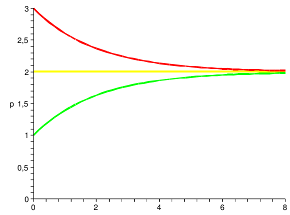
Model of Verhulst – Saturation

- solve initial value problem:

$$\dot{p}(t) = \alpha - \beta p(t), \qquad p(0) = p_0$$

- solution:

$$p(t) = p_\infty + e^{-\beta t}\,(p_0 - p_\infty)\,, \qquad p_\infty = \frac{\alpha}{\beta}$$



Model of Verhulst – Logistic Growth

- saturation model does no longer model exponential growth
- idea: let growth/death **rate** decrease linearly with size of population
- but keep growth/death rate proportional to population size
- leads to differential equation:

$$\dot{p}(t) = (\alpha - \beta p(t))\, p(t)$$

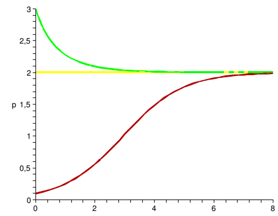
Logistic Growth

- other formulation

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

- solution:

$$p(t) = \frac{\beta}{(1 - e^{-\alpha t}) + \frac{\beta}{p_0} e^{-\alpha t}}$$

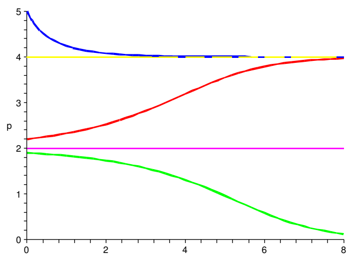


Logistic Growth with Threshold

- extended version of Verhulst's model:

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) \left(1 - \frac{p(t)}{\delta}\right) p(t)$$

- solutions ($\beta = 2, \delta = 4$):



Example – The Passenger Pigeon

- beginning of the 19th century, estimated population in North America: four billion
- hunting diminished its number below a critical threshold (late 1880s)
- the last passenger pigeon died on September 1st, 1914.

(image: Wikipedia)

Part II

Discussion and Analysis of ODE Models

Analysis of ODE Models

Why Analyse a Given Model?

- analytical solutions difficult to compute
- properties of the solution not obvious:
  - shape of solutions?
  - possible steady state?
  - critical points? (species on edge of extinction?)

Methods to Improve Modeling?

- use analysis results to
  - predict failure of the model
  - tune parameters to model a specific situation

Analysing the Slope of a Solution

Example: Model of Malthus

$$\dot{p}(t) = \alpha p(t)$$

- for a physically reasonable solution: $p(t) > 0$
- α decides slope of solution:
  - $\alpha > 0$: growing population (accelerated growth)
  - $\alpha < 0$: receding population (decelerated reduction)

Equilibria and Critical Points

Example: Model of Verhulst (saturation)

$$\dot{p}(t) = \alpha - \beta p(t)$$

- equilibrium: $\dot{p}(t) = 0$
- only, if $p(t) = \frac{\alpha}{\beta}$

Example: Logistic Growth

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

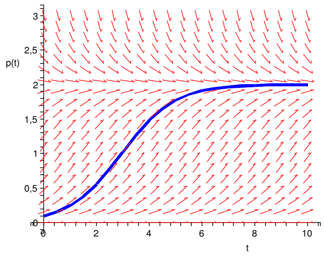
- constant solution, if $p(t) = \beta$ or $p(t) = 0$

Direction Field

plot derivatives vs. time and size of population:

Example: Logistic Growth

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

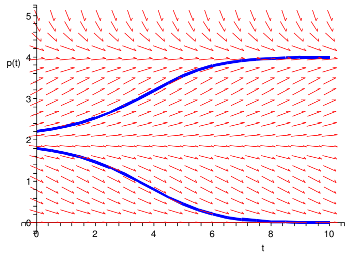


- $p = \beta$ reached for nearly all initial conditions $\rightarrow$ attractive/stable equilibrium/critical point
- $p = 0$ not reached for any other initial conditions $\rightarrow$ repulsive/unstable equilibrium/critical point

Direction Field (2)

Example: Logistic Growth with Threshold

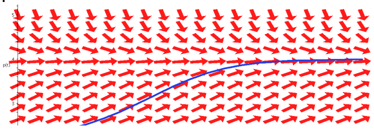
$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) \left(1 - \frac{p(t)}{\delta}\right) p(t)$$



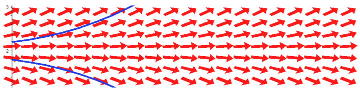
- stable critical points at $p = 0$ and $p = 4$
- unstable critical point at $p = 2$

Identifying Critical Points

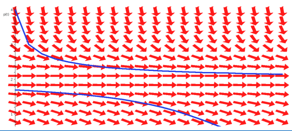
- attractive equilibrium:



- unstable equilibrium



- saddle point



Critical Points – Derivatives

Examine derivatives:

- critical point $p = \bar{p}$
- attractive equilibrium (asymptotically stable):

$$\begin{array}{lll} \dot{p} < 0 & \text{for} & p = \bar{p} + \epsilon \\ \dot{p} > 0 & \text{for} & p = \bar{p} - \epsilon \end{array}$$

- unstable equilibrium:

$$\begin{array}{lll} \dot{p} > 0 & \text{for} & p = \bar{p} + \epsilon \\ \dot{p} < 0 & \text{for} & p = \bar{p} - \epsilon \end{array}$$

- otherwise: saddle point