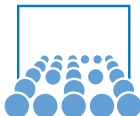


Scientific Computing I

Module 3: Population Modelling – Continuous Models (Parts III and IV)

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Part III: More Than One Species – Systems of ODE

A Linear Model

First Example: Arms Race

Second Example: Competition

A Non-Linear Model

The Non-Linear Competition Model

Predator-Prey

Part IV: Analysis of ODE Models – Two Species

Direction Fields and Critical Points in 2D

- Critical Points in 2D
- 2D Direction Fields
- Summary

Analysis of Systems of ODE

- Homogeneous Systems
- Eigenvalues and Critical Points
- Stability of Linear Systems
- Stability of Non-Linear Systems

Part III

More Than One Species – Systems of ODE

A Linear Model

- similar to Verhulst's saturation model
- additional growth term proportional to other species
- leads to system of differential equations:

$$\dot{p}(t) = b_1 + a_{11}p(t) + a_{12}q(t)$$

$$\dot{q}(t) = b_2 + a_{21}p(t) + a_{22}q(t)$$

- typically:
 - $b_1 > 0, b_2 > 0$ (growth term)
 - $a_{11} < 0, a_{22} < 0$ (saturation)
 - a_{12}, a_{21} ?

First Example: Arms Race

- armament of two (hostile) countries
- our suspicion: $a_{12} > 0$, $a_{21} > 0$

Observation:

- long-time behaviour depends on size of parameters
- steady-state solutions exist
- solutions exist that show unlimited growth

Second Example: Competition

- two species sharing a common natural habitat
- competition: $a_{12} < 0$, $a_{21} < 0$

Observation:

- long-time behaviour depends on size of parameters
- steady-state solutions exist
- some scenarios are physically incorrect!
(negative population size)

A Non-Linear Model

- similar to Verhulst's logistic growth model
- additional growth term proportional to other species
- leads to system of differential equations:

$$\dot{p}(t) = (b_1 + a_{11}p(t) + a_{12}q(t)) p(t)$$

$$\dot{q}(t) = (b_2 + a_{21}p(t) + a_{22}q(t)) q(t)$$

- typically:
 - $b_1 > 0, b_2 > 0$ (growth term)
 - $a_{11} < 0, a_{22} < 0$ (saturation)
 - a_{12}, a_{21} ?

The Non-Linear Competition Model

- two species sharing a common natural habitat
- competition: $a_{12} < 0$, $a_{21} < 0$

Possible Scenarios:

- steady-state
- one species dies out (extinction)
- no obvious nonsense

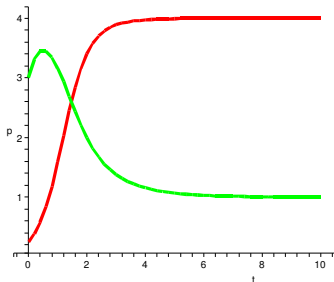
Competition – Steady State

- system of differential equations:

$$\dot{p}(t) = \left(\frac{5}{2} + \frac{\sqrt{3}}{24} - \frac{5}{8}p(t) - \frac{\sqrt{3}}{24}q(t) \right) p(t)$$

$$\dot{q}(t) = \left(\frac{7}{8} + \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{8}p(t) - \frac{7}{8}q(t) \right) q(t)$$

- solution for $p_0 = \frac{1}{4}$, $q_0 = 3$:



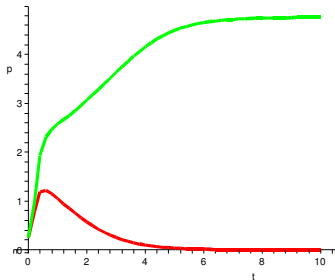
Competition – Extinction

- system of differential equations:

$$\dot{p}(t) = \left(\frac{71}{8} - \frac{23}{12}p(t) - \frac{25}{12}q(t) \right) p(t)$$

$$\dot{q}(t) = \left(\frac{73}{8} - \frac{25}{12}p(t) - \frac{23}{12}q(t) \right) q(t)$$

- solution for $p_0 = \frac{1}{4}$, $q_0 = \frac{1}{4}$:



Predator-Prey

- two species: predator p and prey q
- predator eats prey: $a_{12} > 0$
- prey is eaten by predator: $a_{21} < 0$

Possible Scenarios:

- stable oscillations
- one species dies out (what happens with the other, then?)
- Classical scenario: predator-prey equations by Lotka (1925) and Volterra (1926)

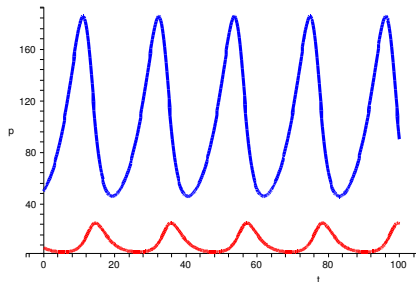
Predator-Prey by Lotka & Volterra

- system of differential equations:

$$\dot{p}(t) = \left(-\frac{1}{2} + \frac{1}{200}q(t)\right)p(t)$$

$$\dot{q}(t) = \left(\frac{1}{5} - \frac{1}{50}p(t)\right)q(t)$$

- solution for $p_0 = 6$, $q_0 = 50$:



Part IV

Analysis of ODE Models – Two Species

Critical Points in 2D

Example: Arms Race

- system of differential equations
- equilibrium: $\dot{p} = 0$, $\dot{q} = 0$

$$\dot{p}(t) = b_1 + a_{11}p(t) + a_{12}q(t) = 0$$

$$\dot{q}(t) = b_2 + a_{21}p(t) + a_{22}q(t) = 0$$

- solution of a linear system of equations:

$$a_{11}p(t) + a_{12}q(t) = -b_1$$

$$a_{21}p(t) + a_{22}q(t) = -b_2$$

- in most cases one critical point
- critical line, if system matrix is singular

Direction Field for a System of ODE

- example: 2D system of differential equations:

$$\dot{p}(t) = b_1 + a_{11}p(t) + a_{12}q(t)$$

$$\dot{q}(t) = b_2 + a_{21}p(t) + a_{22}q(t)$$

- natural extension: 3D plot: t vs. p vs. q

Direction Field for a System of ODE

- example: 2D system of differential equations:

$$\dot{p}(t) = b_1 + a_{11}p(t) + a_{12}q(t)$$

$$\dot{q}(t) = b_2 + a_{21}p(t) + a_{22}q(t)$$

- natural extension: 3D plot: t vs. p vs. q
- 1D direction field for p vs. t or q vs. t not sufficient: what values to choose for q (or p resp.)?
- but: stationary problem \Rightarrow independent of t
- thus: plot directions depending on p and q

2D Direction Field – Arms Race

- system of differential equations:

$$\dot{p}(t) = \frac{3}{2} - p(t) + \frac{1}{2}q(t)$$

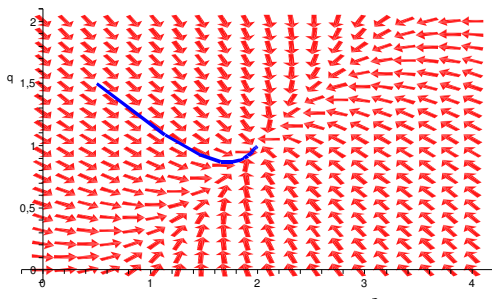
$$\dot{q}(t) = 0 + \frac{1}{2}p(t) - q(t)$$

2D Direction Field – Arms Race

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= \frac{3}{2} - p(t) + \frac{1}{2}q(t) \\ \dot{q}(t) &= 0 + \frac{1}{2}p(t) - q(t)\end{aligned}$$

- direction field – with critical point at (2, 1):



Arms Race – unlimited growth

- system of differential equations:

$$\dot{p}(t) = \frac{1}{2} - \frac{3}{4}p(t) + q(t)$$

$$\dot{q}(t) = -\frac{5}{4} + p(t) - \frac{3}{4}q(t)$$

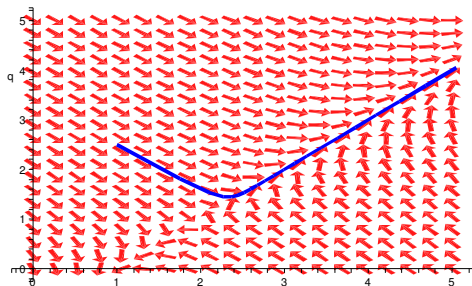
Arms Race – unlimited growth

- system of differential equations:

$$\dot{p}(t) = \frac{1}{2} - \frac{3}{4}p(t) + q(t)$$

$$\dot{q}(t) = -\frac{5}{4} + p(t) - \frac{3}{4}q(t)$$

- direction field – with critical point at (2, 1):



Arms race – the peaceful neighbour

- system of differential equations:

$$\dot{p}(t) = 0 - \frac{3}{4}p(t) + q(t)$$

$$\dot{q}(t) = \frac{5}{2} - p(t) - \frac{3}{4}q(t)$$

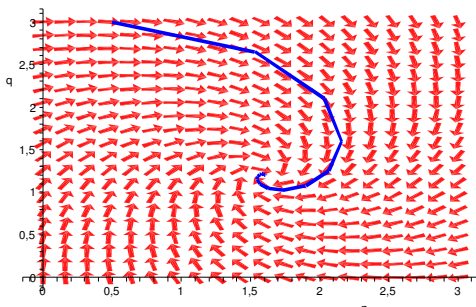
Arms race – the peaceful neighbour

- system of differential equations:

$$\dot{p}(t) = 0 - \frac{3}{4}p(t) + q(t)$$

$$\dot{q}(t) = \frac{5}{2} - p(t) - \frac{3}{4}q(t)$$

- direction field – with critical point at $(\frac{8}{5}, \frac{6}{5})$:



Nonlinear System – Competition

- system of differential equations:

$$\dot{p}(t) = \left(\frac{5}{2} + \frac{\sqrt{3}}{24} - \frac{5}{8}p(t) - \frac{\sqrt{3}}{24}q(t) \right) p(t)$$

$$\dot{q}(t) = \left(\frac{7}{8} + \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{8}p(t) - \frac{7}{8}q(t) \right) q(t)$$

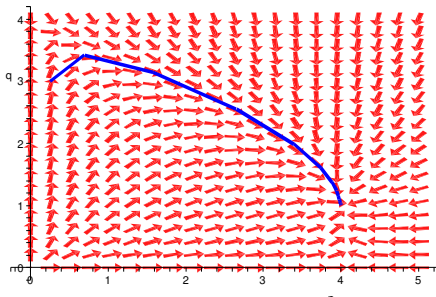
Nonlinear System – Competition

- system of differential equations:

$$\dot{p}(t) = \left(\frac{5}{2} + \frac{\sqrt{3}}{24} - \frac{5}{8}p(t) - \frac{\sqrt{3}}{24}q(t) \right) p(t)$$

$$\dot{q}(t) = \left(\frac{7}{8} + \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{8}p(t) - \frac{7}{8}q(t) \right) q(t)$$

- direction field – critical points at $(4, 1), \dots$:



Nonlinear System – Extinction

- system of differential equations:

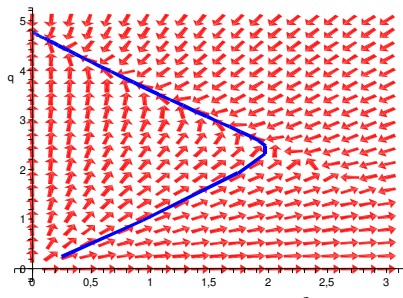
$$\begin{aligned}\dot{p}(t) &= \left(\frac{71}{8} - \frac{23}{12}p(t) - \frac{25}{12}q(t)\right) p(t) \\ \dot{q}(t) &= \left(\frac{73}{8} - \frac{25}{12}p(t) - \frac{23}{12}q(t)\right) q(t)\end{aligned}$$

Nonlinear System – Extinction

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= \left(\frac{71}{8} - \frac{23}{12}p(t) - \frac{25}{12}q(t)\right) p(t) \\ \dot{q}(t) &= \left(\frac{73}{8} - \frac{25}{12}p(t) - \frac{23}{12}q(t)\right) q(t)\end{aligned}$$

- critical points at $(0, 4.76\dots)$, $(4.63\dots, 0)$, \dots :



Lotka & Volterra

- system of differential equations:

$$\dot{p}(t) = \left(-\frac{1}{2} + \frac{1}{200}q(t)\right) p(t)$$

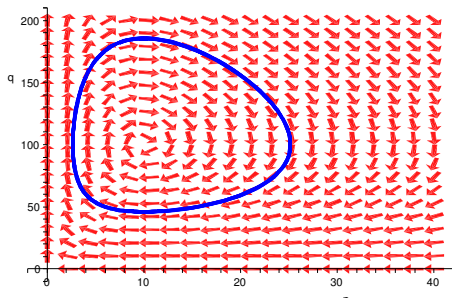
$$\dot{q}(t) = \left(\frac{1}{5} - \frac{1}{50}p(t)\right) q(t)$$

Lotka & Volterra

- system of differential equations:

$$\begin{aligned}\dot{p}(t) &= \left(-\frac{1}{2} + \frac{1}{200}q(t)\right)p(t) \\ \dot{q}(t) &= \left(\frac{1}{5} - \frac{1}{50}p(t)\right)q(t)\end{aligned}$$

- direction field – with critical point at (10, 100):



2D Critical Points – Summary

Different types of critical points in 2D:

- attractive/stable equilibrium
(arms race – steady state)
- unstable equilibrium
- saddle point (arms race – unlimited growth)
- attractive “spiral point” (“peaceful neighbour”)
- unstable “spiral point”
- centre of “rotation” (Lotka-Volterra)

⇒ How to discriminate between these types?

Homogeneous Systems of ODE

Homogeneous System in matrix-vector-notation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n \times n}$
- example: $\mathbf{x}(t) = (p(t), q(t))$

Solutions:

- let \mathbf{x}_λ be an eigenvector: $\mathbf{A}\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda$
- then $\mathbf{x}_\lambda e^{\lambda t}$ is a solution:

$$\mathbf{A}\mathbf{x}_\lambda e^{\lambda t} = \lambda\mathbf{x}_\lambda e^{\lambda t} = \frac{d}{dt}(\mathbf{x}_\lambda e^{\lambda t}) \quad \text{q.e.d.}$$

Eigenvectors and Eigenvalues

Corollaries:

- the solutions of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are linear combinations of the respective eigen-solutions:

$$\mathbf{x}_{\text{hom}}(t) = \sum_{\lambda} a_{\lambda} \mathbf{x}_{\lambda} e^{\lambda t}, \quad a_{\lambda} \in \mathbb{R}$$

- the solutions of the inhomogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$ are

$$\mathbf{x}(t) = -\mathbf{A}^{-1}\mathbf{b} + \mathbf{x}_{\text{hom}}(t)$$

- observation: $\mathbf{x}_{\mathbf{c}} = -\mathbf{A}^{-1}\mathbf{b}$ is a critical point!

Eigenvalues and Critical Points

- the ODE system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{b}$ is solved by

$$\mathbf{x}(t) = \mathbf{x}_c + \sum_{\lambda} a_{\lambda} \mathbf{x}_{\lambda} e^{\lambda t}$$

- \mathbf{x}_c attractive equilibrium,

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_c,$$

only if $e^{\lambda t} \rightarrow 0$ for all eigenvalues λ

- $\lambda \in \mathbb{R} \Rightarrow \lambda < 0$
- $\lambda = \mu + i\nu \Rightarrow \mu < 0$ ($e^{i\nu t} = \cos \nu t + i \sin \nu t$)

Stability of Linear Systems

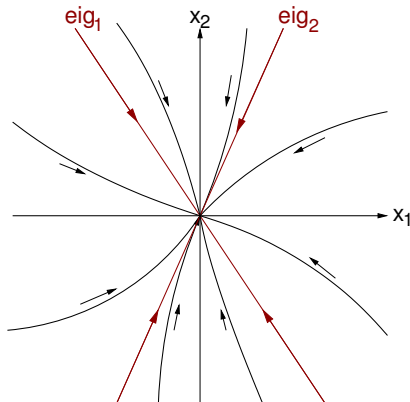
Overview:

eigenval. ($\lambda_j = \mu_j + i\nu_j$)	critical point	stability
real, all $\lambda < 0$	node	stable, attr.
real, all $\lambda > 0$	node	unstable
real, $\lambda_k > 0, \lambda_l < 0$	saddle point	unstable
complex, all $\mu < 0$	spiral point	stable, attr.
complex, all $\mu > 0$	spiral point	unstable
complex, all $\mu = 0$	centre	stable

Stability of 2D Systems

Real Eigenvalues:

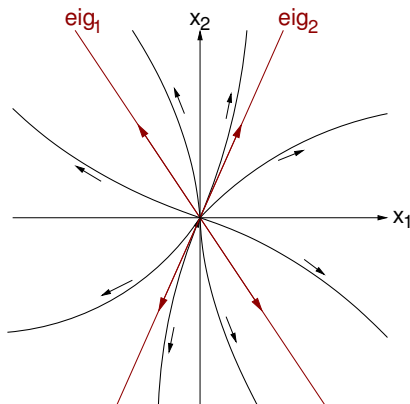
- $\lambda_1 < 0, \lambda_2 < 0$, attractive equilibrium



Stability of 2D Systems

Real Eigenvalues:

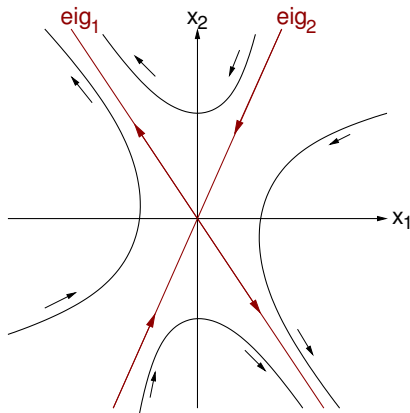
- $\lambda_1 > 0, \lambda_2 > 0$, unstable equilibrium



Stability of 2D Systems

Real Eigenvalues:

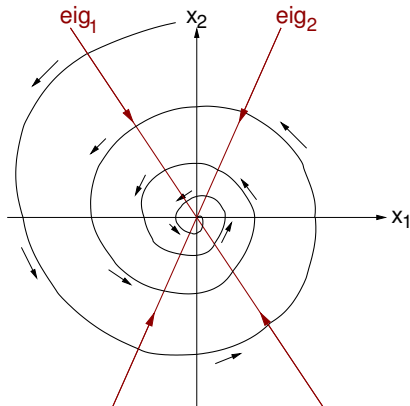
- $\lambda_1 > 0, \lambda_2 < 0$, saddle point



Stability of 2D Systems

Complex Eigenvalues:

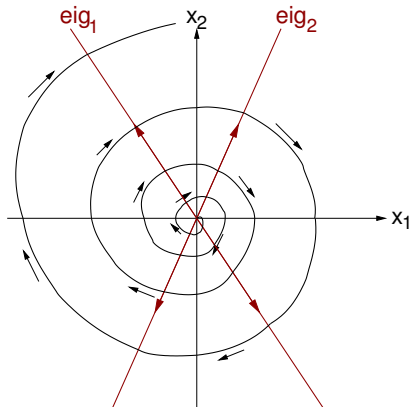
- $\mu_1 < 0, \mu_2 < 0$, spiral point (asympt. stable)



Stability of 2D Systems

Complex Eigenvalues:

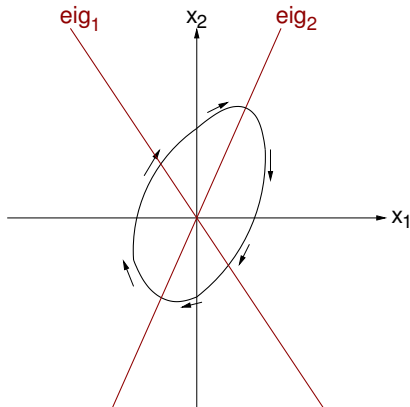
- $\mu_1 > 0, \mu_2 > 0$, spiral point (unstable)



Stability of 2D Systems

Complex Eigenvalues:

- $\mu_1 = \mu_2 = 0$, centre of oscillation



Stability of Non-Linear Systems

- 2D system of ODE:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)),$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ nonlinear

- critical point at \mathbf{x}_c : $f(\mathbf{x}_c) = 0$
- for analysis of critical points: linearization

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \approx \underbrace{f(\mathbf{x}_c)}_{=0} + \mathbf{J}_f(\mathbf{x}_c)(\mathbf{x}(t) - \mathbf{x}_c)$$

- examine eigenvalues of $\mathbf{J}_f(\mathbf{x}_c)$

