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# Worksheet 1
# Problems

**Notation**

In the worksheets you will find different exercises marked by the following symbols:

- **(I)** - marks interactive exercises, they are solved by students during class hours with the help of tutors. Solutions of these exercises might not be provided in a document!

- **(H)** - marks homework exercises, tutors explain and solve these problems during class hours. Solutions are also provided in a document after the tutorial.

- **(H\*)** - marks optional homework exercises for self-study, they are not explained during tutorials. However, solutions are provided after the tutorial.

## Eigenvalues

**Repetition**

The eigenvalue theory can be used to characterise (amongst others) linear systems with respect to the amplification, reduction and frequency of the underlying matrix-based operations. The eigenvalues $\lambda_i$ together with the corresponding eigenvectors $v_i$ for a matrix $A \in \mathbb{R}^{N \times N}$ are all pairs for which hold: $A \cdot v_i = \lambda_i v_i$. It can be shown that the eigenvalues are identical with the roots of the characteristic polynomial $\det(A - \lambda \cdot \mathbb{1})$ where $\mathbb{1}$ denotes the eye-matrix; 'det' represents the *determinant* of the matrix $A - \lambda \cdot \mathbb{1}$. Hence, if $\lambda_i$ is an eigenvalue, then it holds that $\det(A - \lambda_i \cdot \mathbb{1}) = 0$. Similar to the roots of other polynomials, the eigenvalues $\lambda_i$ do not need to necessarily be real values; they might also lie in the space of complex numbers.

Depending on the properties of the matrix $A$, one can find out information about its eigenvalues. In the following, three examples should be given for such properties:

1 A matrix $A \in \mathbb{R}^{N \times N}$ is called *diagonalisable* if it can be written as $A = PDP^{-1}$ with invertible matrix $P \in \mathbb{R}^{N \times N}$ and diagonal matrix $D \in \mathrm{diag}(N)$. In this case, the diagonal matrix contains the eigenvalues $\lambda_i$ on its diagonal and the columns of $P$ represent the corresponding eigenvectors.

2 A matrix $A \in \mathbb{R}^{N \times N}$ is called symmetric if it holds $A_{ij} = A_{ji}$ for all $i, j = 1, ..., N$. If a matrix is symmetric, all eigenvalues are real values.

3 A matrix $A \in \mathbb{R}^{N \times N}$ is called *positive definite* if $x^\top A x > 0$ for all vectors $x \in \mathbb{R}^N \setminus \{\vec{0}\}$ and *positive semi-definite* if $x^\top A x \geq 0$ for all $x \in \mathbb{R}^N \setminus \{\vec{0}\}$. Analogous definitions hold for *negative (semi-)definiteness*. For symmetric matrices the following equivalence is true

$$\text{matrix } A \text{ is positive definite} \Leftrightarrow \text{all eigenvalues are positive}$$

In the following exercises, we will practise the computation of eigenvalues and the characterisation of matrix-based systems in terms of their eigenvalues.

**Remarks & Hints**

- You may use the following rule to compute the determinant of a $2 \times 2$ matrix:

$$\det \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11}a_{22} - a_{12}a_{21} \tag{1}$$

- You may use the following rule to compute the determinant of a $3 \times 3$ matrix:

$$\det \left( \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \tag{2}$$

**(I) Exercise 1: Direct Computation of Eigenvalues**

Consider the matrices

$$A := \begin{bmatrix} -6 & -14 & -12 \\ 4 & 9 & 6 \\ 1 & 2 & 3 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

(a) Compute the eigenvalues of the matrices.

(b) Compute the eigenvectors of $B$.

(c) Which properties does the matrix $B$ have?

## Population Models

**(I) Exercise 2: Web-stores**

Two web-stores were successfully selling one product for a similar price. To increase their income they decided to change the prices. Complete the following tasks to find out if their strategies worked fine.

(a) The first web-store decided to enhance the selling of this product by gradually decreasing the price. Each week they would change the price by taking a sum of 50% of their current price and 40% of their competitors current price. Write down a recursive formula to find the price $a_{n+1}$ at week $n + 1$, if the price at week $n$ is $a_n$ and the competitors' price is fixed to $b$. What value will the price approach after many weeks? In other words, what is the stationary solution of the recursive formula?

(b) The second web-store decided to increase the price of the product. They would make a weekly price adjustment by taking 50% of their current price and 60% of their competitors current price. Find a recursive formula for the price $b_{n+1}$, if the current price is $b_n$ and the competitors' price is fixed to $a$. What value will the price approach in the long run?

(c) The web-stores started applying their strategies simultaneously. Find the matrix $A$ in the recursive relation

$$\begin{pmatrix} a(n+1) \\ b(n+1) \end{pmatrix} = A \cdot \begin{pmatrix} a(n) \\ b(n) \end{pmatrix} \tag{4}$$

(d) After some time the two web-stores noticed that something was going wrong with the product prices. Find the eigenvalues of matrix $A$ and explain what could go wrong.

## (H) Exercise 3: Rabbits and Fibonacci Numbers

Let's consider the Fibonacci model for the evolution of rabbits. Each pair is assumed to have one pair of children each year (one male and one female rabbit). In their first year, the young rabbits do not have children. After the first year, they will also give birth to one pair of rabbits each year.

(a) Let $Y$ denote the "young" rabbits and $G$ the "grown-up" rabbits. Model the evolution of the rabbits by a recursive scheme and a respective linear relationship between the rabbits $Y(n), Y(n+1), G(n), G(n+1)$ of subsequent years $n, n+1$:

$$\begin{pmatrix} Y(n+1) \\ G(n+1) \end{pmatrix} = A \cdot \begin{pmatrix} Y(n) \\ G(n) \end{pmatrix} \tag{5}$$

with matrix $A \in \mathbb{R}^{2 \times 2}$.

(b) Which properties does $A$ have? Compute the eigenvalues and eigenvectors of $A$!

(c) Assume an initial rabbit population $(Y(0), G(0))^\top := (0, 1)^\top$. How can we describe the evolution of the rabbits in terms of eigenvectors? How can we easily estimate the population in the year 20 by only considering the eigenvalues and eigenvectors of the system?
Hint: decompose the initial population $(Y(0), G(0))^\top$ into its eigenvector contributions and compute their evolution separately.

(d) Assume that each year $p$ % of the grown-up rabbits and $q$ % of the young rabbits die due to a big fat wolf in their forest. How can you include this assumption into the recursive

model from above? How do the eigenvalues of the update matrix $A$ change in this case? Remark: for the sake of simplicity, assume that dying and giving birth is a strictly sequential process ;-), i.e. the respective percentage of rabbits dies first and the remaining rabbits give birth to new pairs of rabbits.

## (H*) Exercise 4: Rates and Calculation of Interest

Congratulations! You are just about to open a new bank account. To open it, you initially invest $K(n = 0)$ euros, that is you start with $K(n = 0)$ euros in the year $n = 0$. After each year, you first obtain an interest rate of $p$ % onto your current savings. Besides, you are obliged to add another $J$ euros each year onto your current account.

(a) Try to find a model for your bank account which–based on a recursive formula–computes your savings $K(n + 1)$ in the $(n + 1)$-th year from the savings $K(n)$.

(b) Which value can be considered to be an eigenvalue in our recursive expression? Which quantities affect the eigenvalue and what happens to your savings when you modify them?

(c) Convert the recursive relation from (a) into a non-recursive expression.

(d) The people from the bank cheated on you. Though they first announced that the bank account is for free, you suddenly need to pay $n$ euros in the year $n$, starting in the very first year (hence, only the first year was free)! Include the arising costs into the recursive expression from (a).

(e) How do the costs of $n$ euros in year $n$ enter the non-recursive formula for your savings?

(f) What can you buy from your saved money in 10 years, assuming an interest rate $p = 0.05$, an initial payment of $K(n = 0) = 50$ euros and annual investments of $J = 50$? A new notebook ($\sim$ 1000 Euros), the latest iPad ($\sim$ 800 Euros), or the latest iPhone ($\sim$ 650 Euros)?