Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel
M.Sc. D. Jarema

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Worksheet 12
Problems

Finite Element Methods

(H) Exercise 1: Convection-Diffusion Equations

Consider the convection-diffusion equation for temperature transport in a fluid which moves at constant velocity $v \in \mathbb{R}$:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $D \in \mathbb{R}^+$ denotes the diffusion constant of the fluid. The problem shall be solved on the unit interval with homogeneous Dirichlet conditions.

(a) Derive the weak formulation of the equation. Discretize in space by piecewise linear hat functions. Derive the semi-discrete set of equations and compute all coefficients. How can we categorize this set of equations?

(b) Perform mass lumping to facilitate the time discretization. Approximate the mass matrix $M_{ij} := \int \varphi_i \varphi_j dx$ by a diagonal matrix $\tilde{M}_{ij}$,

$$\tilde{M}_{ij} := \begin{cases} \sum_j M_{ij} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Use the explicit Euler method to subsequently discretize the problem in time.

(c) Solve the problem with python for $D = 1.0$, $v = 1.0$, a mesh size $h = 1/10$ and a time step $\tau = 0.002, 0.02$. Use initial conditions $T = 1$ inside the domain (and homogeneous Dirichlet conditions at the boundaries).

(H) Exercise 2: Reference Elements

In the following, we want to compute the mapping from an arbitrary triangle E onto a reference triangle E_{ref}, cf. Figure 1. This can be useful in several contexts, for example to simplify the integration procedures in the FE method or to prove error estimates for the respective finite elements and their basis functions.

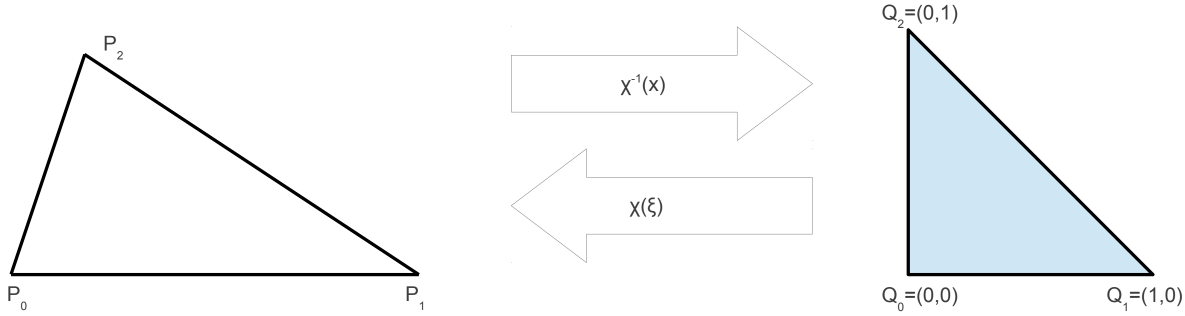

Figure 1: Coordinate mapping between a triangle of arbitrary shape (left) and a reference triangle (right).

(a) Define a transformation $\chi(\xi)$ which maps the coordinates ξ within the reference triangle E_{ref} (triangle on the right in Figure 1) onto the triangle E (triangle on the left in Figure 1). Use arbitrary coordinates P_0, P_1, $P_2 \in \mathbb{R}^2$ to define the transformation.

(b) Denote the corners of the reference triangle by $Q_0 = (0,0)^\top$, $Q_1 = (1,0)^\top$, $Q_2 = (0,1)^\top$. Define linear functions $\Phi_i(\xi)$, $i = 0,1,2$, on the reference triangle such that $\Phi_i(Q_j) = \delta_{ij}$, that is

$$\Phi_i(Q_j) := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{3}$$

Compute the mass matrix $M_{ij}^{\text{ref}} := \int_{E_{\text{ref}}} \Phi_i(\xi)\Phi_j(\xi)d\xi$ of the reference element; you may use python for this purpose.

(c) Use the variable substitution to derive a formula which evaluates the mass matrix $M_{ij} = \int_E \phi_i(x)\phi_j(x)dx$ for an arbitrary triangle and its respective basis functions $\phi_i(x)$. The formula may only make use of the transformation $\chi(\xi)$ and the mass matrix M^{ref} of the reference triangle.

(d) Validate your formula from task (c) by computing the mass matrix of the reference element from Worksheet 11, Exercise 2. You may use python for this purpose.

(H*) Exercise 3: Circus Tent

In order to analyze the stability of a circus tent, a two-dimensional simulation of the forces that act onto the tent construction is required. The tent construction is shown in Figure 2 on the left: from the middle of the tent, *spanning rods* branch out to the outer end of the circular tent. Each spanning rod has a length R. The spanning rods are homogeneously distributed; every pair of neighboring spanning rods is separated by an angle α. The ends of two spanning rods are held together by a *connecting rod*. Two spanning rods and one connecting rod form a triangular finite element. The element E_0 is chosen as reference element $E := E_0$; it is depicted in Figure 2 on the right.

(a) Determine linear basis functions $\Phi_i(x,y)$, $i = 0,1,2$, on the reference element such that $\Phi_i(Q_j) = \delta_{ij}$ for the points $Q_0 = (0,0)$, $Q_1 = (R,0)$, $Q_2 = (L,H)$. The basis functions may depend on L, R and H, respectively.

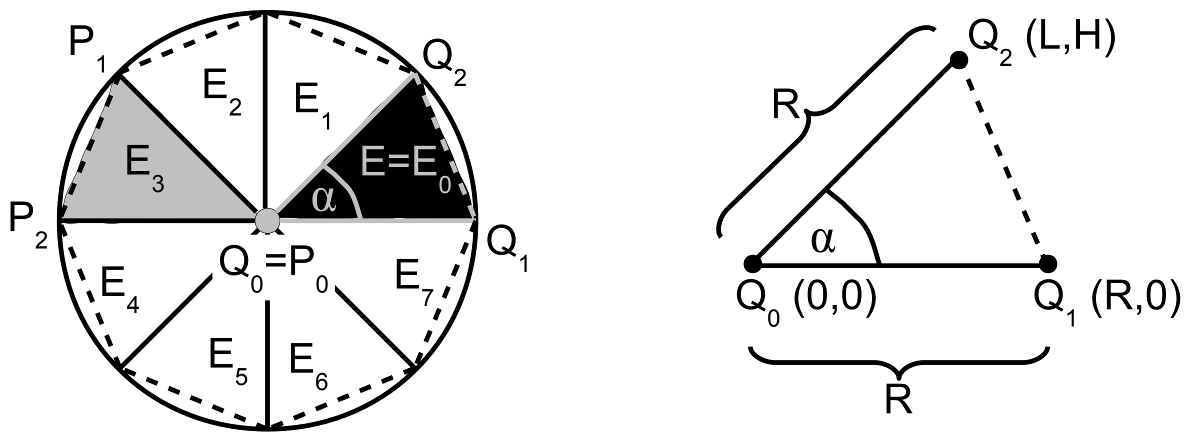

Figure 2: Left: circus tent. The tent is spanned by several rods branching out from the very middle. The rods are homogeneously distributed and are separated by an angle α. Additional rods (denoted by dashed lines) connect the outer ends of the spanning rods. Right: zoom into one of the elements that is spanned by two spanning rods and one connecting rod.

(b) Compute the entry

$$A_{00}^{\text{ref}} := \int_E \nabla\Phi_0 \cdot \nabla\Phi_0. \tag{4}$$

The final expression for A_{00}^{ref} should only depend on the angle α, i.e. $A_{00}^{\text{ref}} = A_{00}^{\text{ref}}(\alpha)$. It should not depend on L, R or H anymore.

(c) Determine a linear transformation rule $\chi^{(k)} : E \to E_k$ which maps the reference element $E = E_0$ onto any of the other elements E_k. The transformation should thus satisfy $\chi^{(k)}(Q_i) = P_i$, $i = 0,1,2$, where P_i corresponds to the respective vertex coordinates Q_i in the same consistent elementwise numbering scheme for element E_k (cf. Figure 2 on the left: the local vertex numbering is shown for elements E_0 and E_3.). The enumeration of the elements E_k is accomplished counter-clockwise starting from the reference element $E = E_0$ as illustrated in Figure 2 on the left.

Give an analytical expression for $\chi^{(k)}$ and show that the mass matrix entries (arising from Poisson-like PDE problems) are equal for all elements E_k, that is

$$\int_{E=E_0} \phi_i^{(0)}\phi_j^{(0)} = \int_{E_1} \phi_i^{(1)}\phi_j^{(1)} = \ldots = \int_{E_k} \phi_i^{(k)}\phi_j^{(k)} = \ldots \tag{5}$$

where $\phi_i^{(k)}$ denotes the i-th linear basis function ($i = 0,1,2$) which is locally defined on each element (k) as $\phi_i^{(k)}(P_j) = \delta_{ij}$.

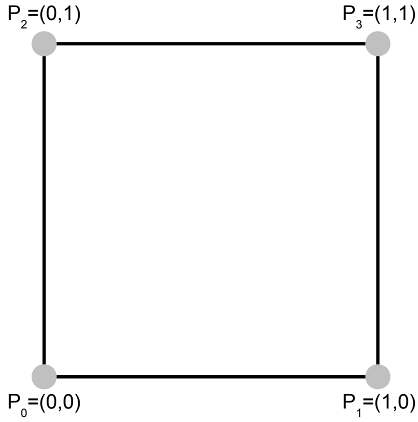

Figure 3: Cartesian grid cell for the application of bilinear ansatz functions.

(H*) Exercise 4: Stationary Convection-Diffusion Equations

The following differential equation for an unknown function $u(x, y)$ is defined on a square, $\Omega := (0, a) \times (0, b)$, with homogeneous Dirichlet conditions on all boundaries of the square:

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + q(x, y) \tag{6}$$

where $q(x, y)$ denotes a (known) source term.

We want to solve the problem on a Cartesian grid using the standard Galerkin procedure. For this purpose, we introduce locally *bilinear* basis functions which are defined on the reference element E given in Figure 3. The basis functions on the local reference element are given by

$$\begin{aligned} \varphi_0(x, y) &= (1 - x)(1 - y), \\ \varphi_1(x, y) &= x(1 - y), \\ \varphi_2(x, y) &= (1 - x)y, \\ \varphi_3(x, y) &= xy. \end{aligned} \tag{7}$$

For the corners P_0,..., P_3 of the reference element, cf. Figure 3, this yields $\varphi_i(P_j) = \delta_{ij}$ similar to the case of using piecewise linear basis functions on triangles.

(a) Derive the weak formulation of equation (6) for test functions $\psi(x, y)$ which belong to some function space V, $\psi(x, y) \in V$. No discretization of the function space V is required. Use integration by parts to transform the second-order derivative. Give a brief explanation why this transformation is helpful.

(b) Discretize your weak formulation using the basis functions $\varphi_i(x, y)$. Re-write the system in matrix-vector form $C \cdot \vec{u} = A \cdot \vec{u} + M \cdot \vec{q}$ with matrices

- C describing convection
- A describing diffusion

- M invoking sources terms.

Give a definition for each matrix entry C_{ij}, A_{ij} ,M_{ij}.

(c) Compute the contributions of the matrices A_{33} and C_{33} on the reference element.