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Worksheet 4
Problems

Continuous Models: Ordinary Differential Equations

(H) Exercise 1: Direction Fields for ODE

Consider the ordinary differential equation

$$\frac{dy(t)}{dt} = \lambda y(t)^2 + \mu y(t) - \nu$$

with real constants $\lambda, \mu, \nu \geq 0$.

(a) For $\lambda = 1$, $\mu = 0$, $\nu = 1$, compute the critical points, compute their characteristics (stable, unstable, saddle point) and sketch the respective direction field for $t \in [0,4]$, $y \in [-2,2]$.

(b) Write a python script which sketches the direction fields of the ODE from above for arbitrary choices of λ, μ, ν.

(c) Compute the critical points of the ODE and characterize them using exemplary direction field plots of the python script, i.e. for each relevant parameter combination, choose at least one parameter set, visualize the underlying direction field and determine the characteristics of the critical points.

(H) Exercise 2: Direction Fields for a System of ODEs

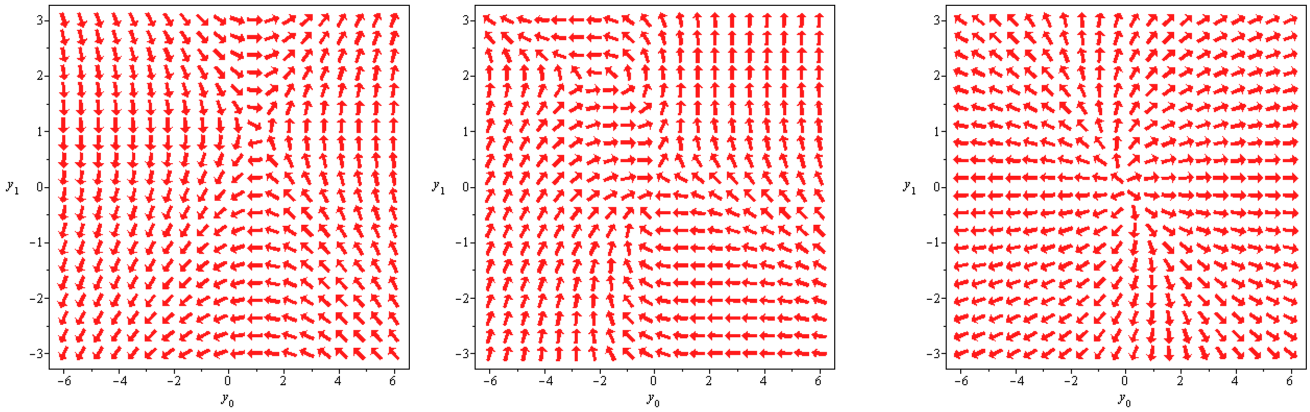
Direction fields can be used to determine the characteristics of ODEs.

(a) Three different direction fields for different systems of two ODEs

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = f(t, y_0(t), y_1(t)) \tag{1}$$

are given in Figure 1. Which of the direction fields (a), (b), (c) belongs to

1. a linear homogeneous system of ODEs



(a) (b) (c)

Figure 1: Different direction fields for systems of two ODEs.

2. a linear inhomogeneous system of ODEs
3. a non-linear system of ODEs?

Give a short explanation for each decision and specify the number and type of equilibrium points of the direction fields that belong to 1. and 2.

(b) A particular system of two differential equations is given by

$$\begin{aligned} \frac{dy_0(t)}{dt} &= 3y_0(t) + y_1(t) - 2 \\ \frac{dy_1(t)}{dt} &= 2y_0(t) + 2y_1(t) - 1. \end{aligned} \tag{2}$$

Write the system of ODEs from Equation (2) in matrix-vector form, determine the equilibrium points of the system and specify their type via analytical computation.

(I) Exercise 3: Population Modeling - Two Species

Given are four different scenarios of a two-species population model. Figure 2 shows the direction fields for these four scenarios. Figure 3 shows the respective solutions.

(a) State which solution plots belongs to which direction fields. Draw the evolution trajectories of the population sizes p and q into Figure 2.

(b) One of the four scenarios uses a non-linear model; the other three scenarios are from a linear model. State which solution plot and direction field belong to the non-linear model. What is the reason of your choice?

(c) All four scenarios have a critical point for the same values of p and q. State what type of critical point it is for each of the models.

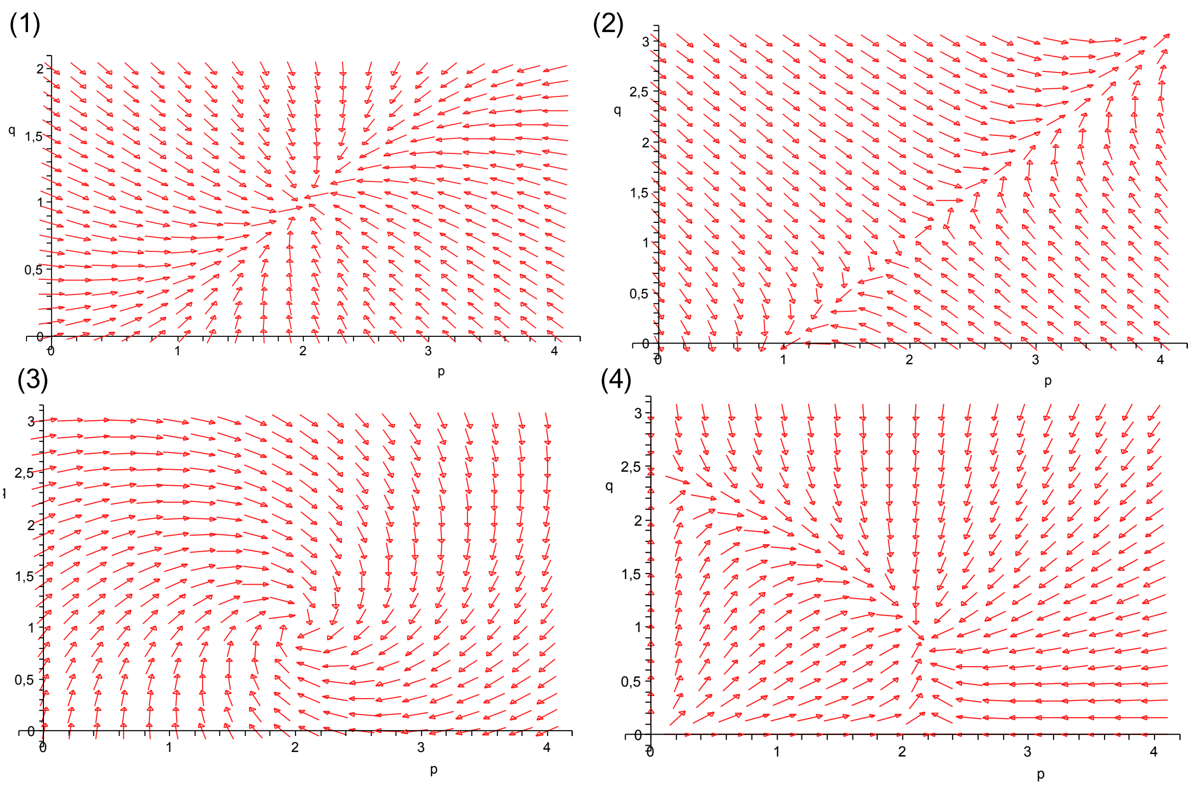


Figure 2: Direction fields for population models with two species. Population p is plotted along the horizontal axis and population q is plotted along the vertical axis.

(H*) Exercise 4: Exponential Function for Matrices

Similar to scalars, the exponential function can be extended to matrices. It is defined for a matrix $A \in \mathbb{R}^{N \times N}$ as

$$exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

and can be used to analytically solve systems of ordinary differential equations.
Consider the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(a) Compute A^2, A^3, A^4 and A^5 and derive a general formula for A^{2k} and A^{2k+1}, $k \in \mathbb{N}$.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}^{2 \times 2}$, $f(t) := exp(At)$. Show that

$$f(t) = I \cdot \cos(t) + A \cdot \sin(t)$$

where I is the identity matrix.
Hint: Use the results from (a) and consider the series representation of the trigonometric functions.

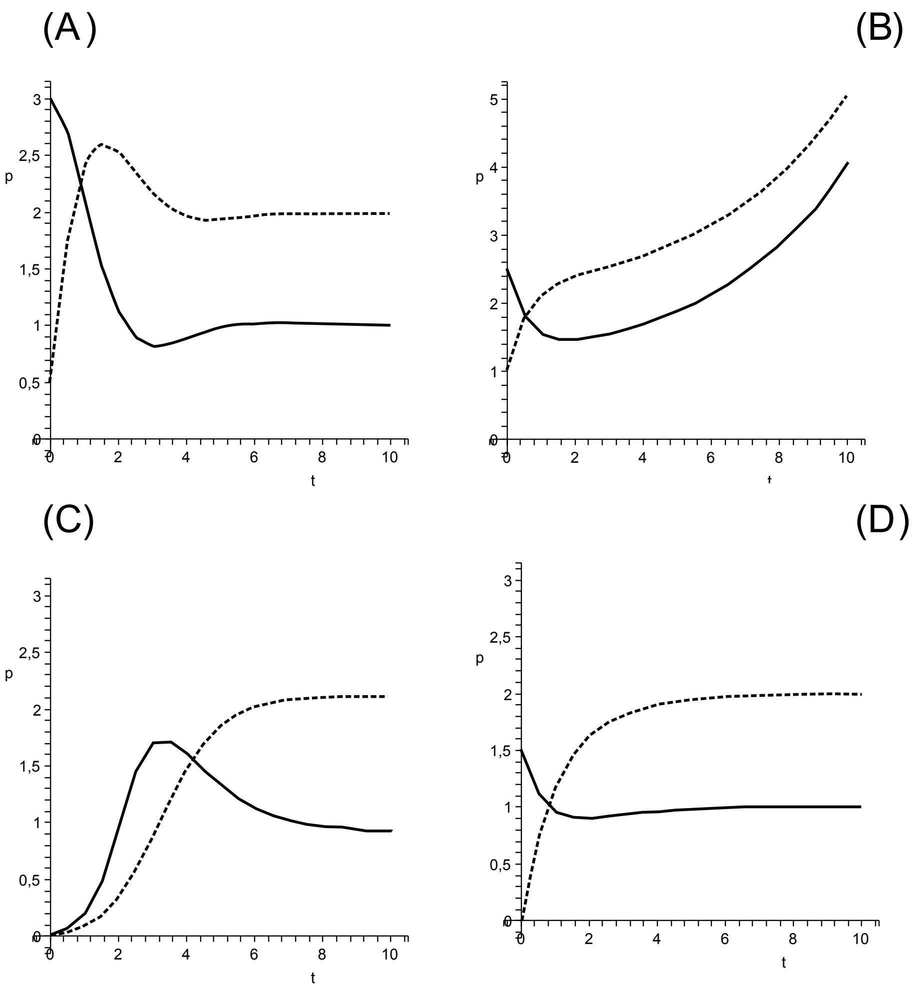


Figure 3: Solution examples for population models with two species. The dash lines show evolution of population p and the solid lines of population q.