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Scientific Computing
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Worksheet 4
Sample Solutions

Continuous Models: Ordinary Differential Equations

(H) Exercise 1: Direction Fields for ODE

Consider the ordinary differential equation

$$\frac{dy(t)}{dt} = \lambda y(t)^2 + \mu y(t) - \nu$$

with real constants $\lambda, \mu, \nu \geq 0$.

(a) For $\lambda = 1$, $\mu = 0$, $\nu = 1$, compute the critical points, compute their characteristics (stable, unstable, saddle point) and sketch the respective direction field for $t \in [0, 4]$, $y \in [-2, 2]$.

(b) Write a python script which sketches the direction fields of the ODE from above for arbitrary choices of λ, μ, ν.

(c) Compute the critical points of the ODE and characterize them using exemplary direction field plots of the python script, i.e. for each relevant parameter combination, choose at least one parameter set, visualize the underlying direction field and determine the characteristics of the critical points.

Solution:

(a) With the specified coefficients the ODE takes form

$$\frac{dy(t)}{dt} = f(y) = y(t)^2 - 1,$$

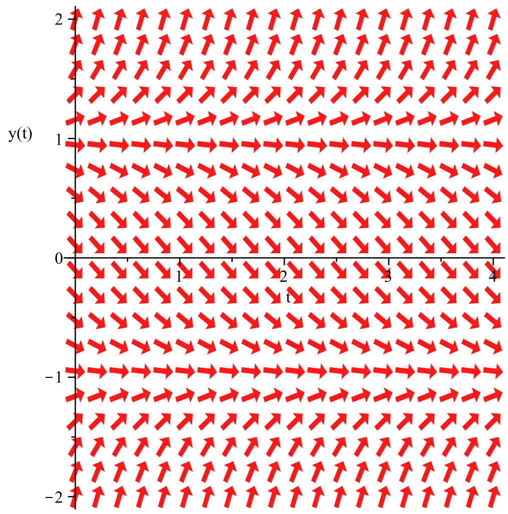
The critical points of this equation are $y_{1,2}(t) = \pm 1$.

To analyze the behavior of the system, we compute the derivatives of the right hand side of the ODE ($df(y)/dy = 2y$) at the critical points.

- For $y_1(t) = 1$ we obtain $df(y)/dy = 2 > 0$. According to the definition from the lecture, we see that $y_1(t)$ must be an unstable equilibrium point.

- For $y_2(t) = -1$ we obtain $df(y)/dy = -2 < 0$. According to the definition from the lecture, we see that $y_2(t)$ must be an attractive equilibrium point.

The sketch of the direction field looks as follows:



(b) See ws4_ex1.py

(c) The critical points evolve for $dy(t)/dt \stackrel{!}{=} 0$. First investigate the case that $\lambda \neq 0$:

$$\begin{aligned} \frac{dy(t)}{dt} &= \lambda y(t)^2 + \mu y(t) - \nu &\stackrel{!}{=}\; & 0 \\ \Leftrightarrow \qquad & y_{1,2}(t) &=\; & \frac{-\mu \pm \sqrt{\mu^2 + 4\lambda\nu}}{2\lambda} \end{aligned}$$

Hence, for $\mu = 0$ and $\nu = 0$, we have only one critical point at $-\mu/(2\lambda) \stackrel{\mu=0}{=} 0$. If we consider for example the set $\lambda = 1$, $\mu = \nu = 0$ and plot the direction field over $t = 0..20$, $y = -1..1$, we can observe that the arrows in the lower part ($y < 0$) tend towards $y(t) = 0$ whereas the direction field points away from the critical point in the upper part of the graph ($y > 0$). We conclude that the respective critical point is a saddle point.

For all other configurations of μ, ν, we obtain two critical points of the system. Let's consider the case $\lambda = 5$, $\mu = 4$, $\nu = 1$. The arising critical points are given by $y_1(t) = -1$ and $y_2(t) = \frac{1}{5}$. Plotting the arising direction fields over $t = 0..5$, $y = 0..0.8$, we see that $y_2(t)$ is unstable. For $t = 0..1$, $y = -2..0$, one can observe that the critical point $y_1(t)$ is asymptotically stable (attractive).

Now, let's turn to the case $\lambda = 0$. If $\lambda = 0$, we only have a linear equation system to solve for $y(t)$ and obtain:

$$\begin{aligned} \frac{dy(t)}{dt} &= 0 \cdot y(t)^2 + \mu y(t) - \nu &\stackrel{!}{=}\; & 0 \\ \Leftrightarrow \qquad & y(t) &=\; & \frac{\nu}{\mu} \end{aligned}$$

Here, we need to assume that $\mu \neq 0$. Using the python script, we see that this case corresponds to a stable or unstable point (for example, consider the direction field plot for $\lambda = 0$, $\mu = \pm 1$, $\nu = 1$, $t = 0..20$, $y = -5..5$). Finally, we consider the case where $\lambda = 0$ and $\mu = 0$. If $\nu \neq 0$, the equality $dy/dt \overset{!}{=} 0$ can never be reached. If $\nu = 0$, the respective equality is always fulfilled for all solutions since the arising solution is the constant curve, $y(t) = y_0$ with $y_0 := y(t = 0)$.

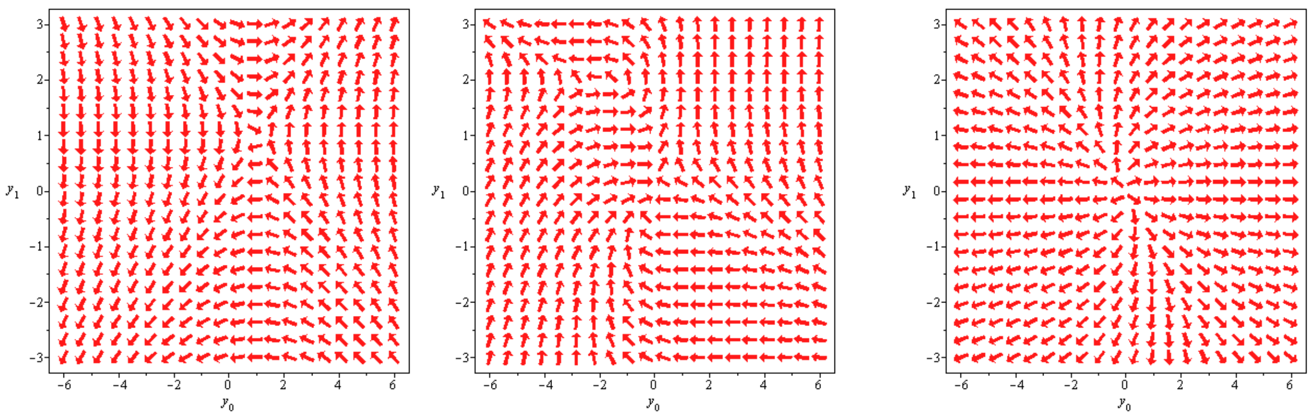
(H) Exercise 2: Direction Fields for a System of ODEs

Direction fields can be used to determine the characteristics of ODEs.

(a) Three different direction fields for different systems of two ODEs

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = f(t, y_0(t), y_1(t)) \tag{1}$$

are given in Figure 1. Which of the direction fields (a), (b), (c) belongs to



(a) (b) (c)

Figure 1: Different direction fields for systems of two ODEs.

1. a linear homogeneous system of ODEs
2. a linear inhomogeneous system of ODEs
3. a non-linear system of ODEs?

Give a short explanation for each decision and specify the number and type of equilibrium points of the direction fields that belong to 1. and 2.

(b) A particular system of two differential equations is given by

$$\begin{aligned} \frac{dy_0(t)}{dt} &= 3y_0(t) + y_1(t) - 2 \\ \frac{dy_1(t)}{dt} &= 2y_0(t) + 2y_1(t) - 1. \end{aligned} \tag{2}$$

Write the system of ODEs from Equation (2) in matrix-vector form, determine the equilibrium points of the system and specify their type via analytical computation.

Solution:

(a) The linear homogeneous system of ODEs must have an equilibrium state at $(0,0)$. The only graph with this property is graph (c). It is an unstable equilibrium point and thus must have two eigenvalues > 0, cf. lecture slides. The inhomogeneous system may have a line or a single point as equilibrium state; this state can, however, be at any location (not necessarily at $(0,0)$). A known characteristics for a respective graph is shown in graph (a) which has an equilibrium point at $\sim (1,1)$ and represents a saddle point. The only system left over is graph (b). It does neither show spiral point, saddle point or any other well-known kind of equilibrium characteristics. It must therefore stem from the non-linear case.

(b) We re-write the system of ODEs in matrix-vector form:

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix}}_{:=A \in \mathbb{R}^{2\times 2}} + \underbrace{\begin{pmatrix} -2 \\ -1 \end{pmatrix}}_{:=b} \tag{3}$$

The equilibrium points arise from $A \cdot y + b \overset{!}{=} 0$. Solving the linear system yields $y_0 = \frac{3}{4}$, $y_1 = -\frac{1}{4}$. The eigenvalues of A determine the type of the equilibrium point. We need to compute the eigenvalues via

$$\det A - \lambda I = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = ... = \lambda^2 - 5\lambda + 4 \overset{!}{=} 0 \tag{4}$$

Solving the quadratic equation yields $\lambda_0 = 1 > 0$, $\lambda_1 = 4 > 0$. Since both eigenvalues are real and bigger than zero, the equilibrium point is unstable.

(I) Exercise 3: Population Modeling - Two Species

Given are four different scenarios of a two-species population model. Figure 2 shows the direction fields for these four scenarios. Figure 3 shows the respective solutions.

(a) State which solution plots belongs to which direction fields. Draw the evolution trajectories of the population sizes p and q into Figure 2.

(b) One of the four scenarios uses a non-linear model; the other three scenarios are from a linear model. State which solution plot and direction field belong to the non-linear model. What is the reason of your choice?

(c) All four scenarios have a critical point for the same values of p and q. State what type of critical point it is for each of the models.

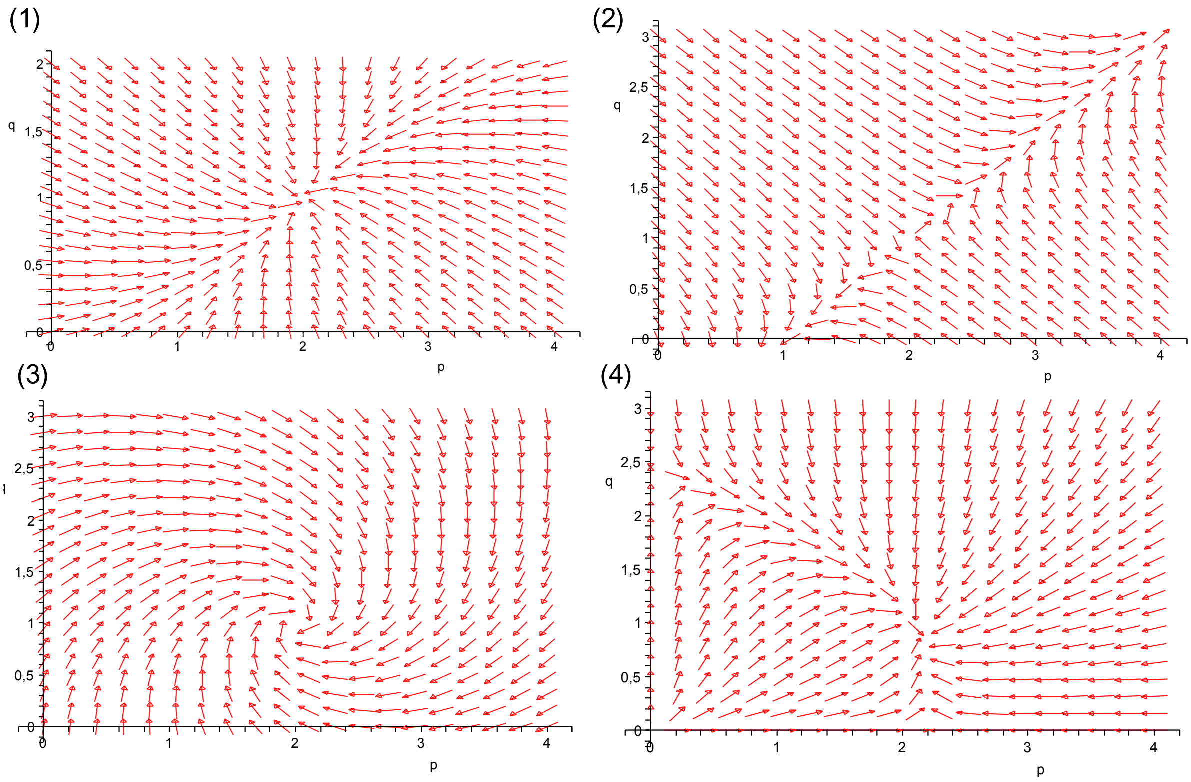


Figure 2: Direction fields for population models with two species. Population p is plotted along the horizontal axis and population q is plotted along the vertical axis.

Solution:

(a) `(B)` belongs to `(2)` – no stable critical point in the direction field, but simultaneous growth of both p and large t.

`(A)` belongs to `(3)` – stable critical point, i.e. equilibrium for $t \to \infty$ at $p = 2$ and $q = 1$, change of sign in the slope of p and q due to spiral point.

`(C)` belongs to `(4)` – critical point at equilibrium of the solution plots, not-monotonic behavior of q if starting from $(0, 0)$.

`(D)` belongs to `(1)` – critical point at equilibrium of the solution plots, monotonic growth of p and decrease of q towards equilibrium if starting from (0,1.5).

For the evolution trajectories see Figure 4.

(b) `(4)` belongs to a non-linear model as the two eigenvalues of the system depend on p and q.

(c) `(1)` stable equilibrium – no spiral form of direction field, arrows form all directions pointing towards the critical point.

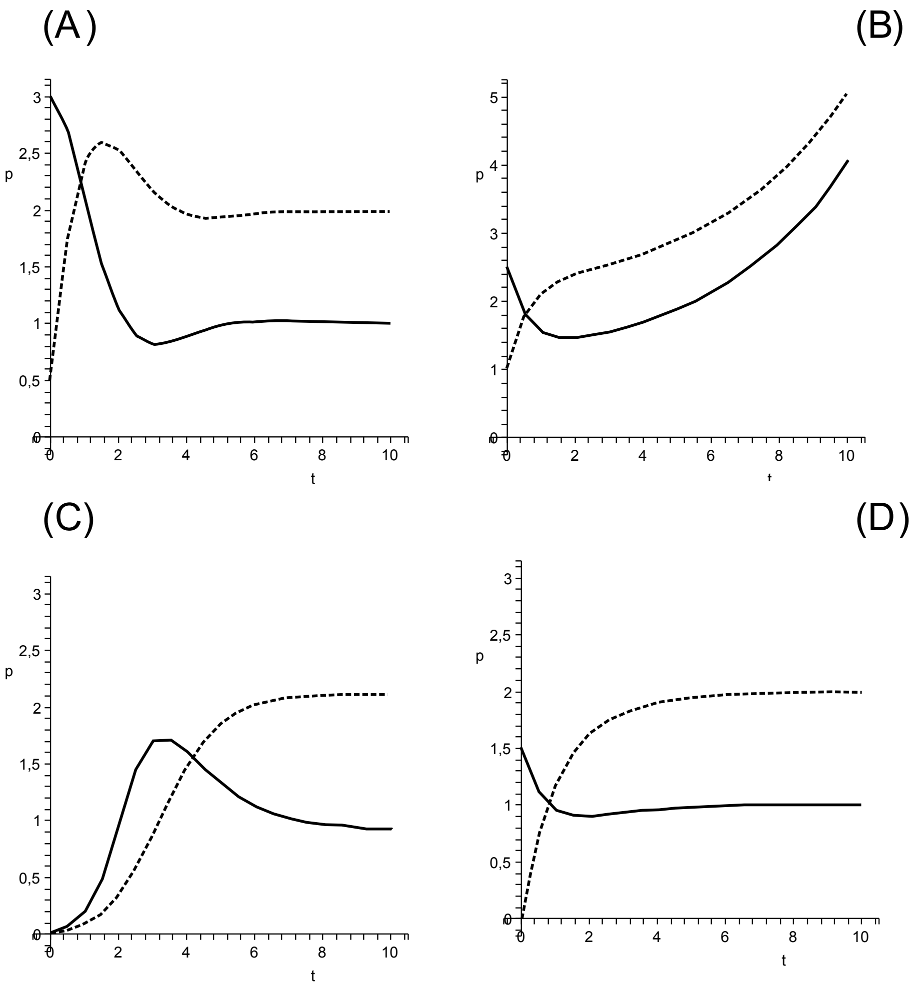


Figure 3: Solution examples for population models with two species. The dash lines show evolution of population p and the solid lines of population q.

(2) saddle point – attracting solutions only along a line, unstable in another line direction.

(3) attractive spiral point – spiral form of direction fields, all arrows pointing towards the critical point.

(4) stable equilibrium.

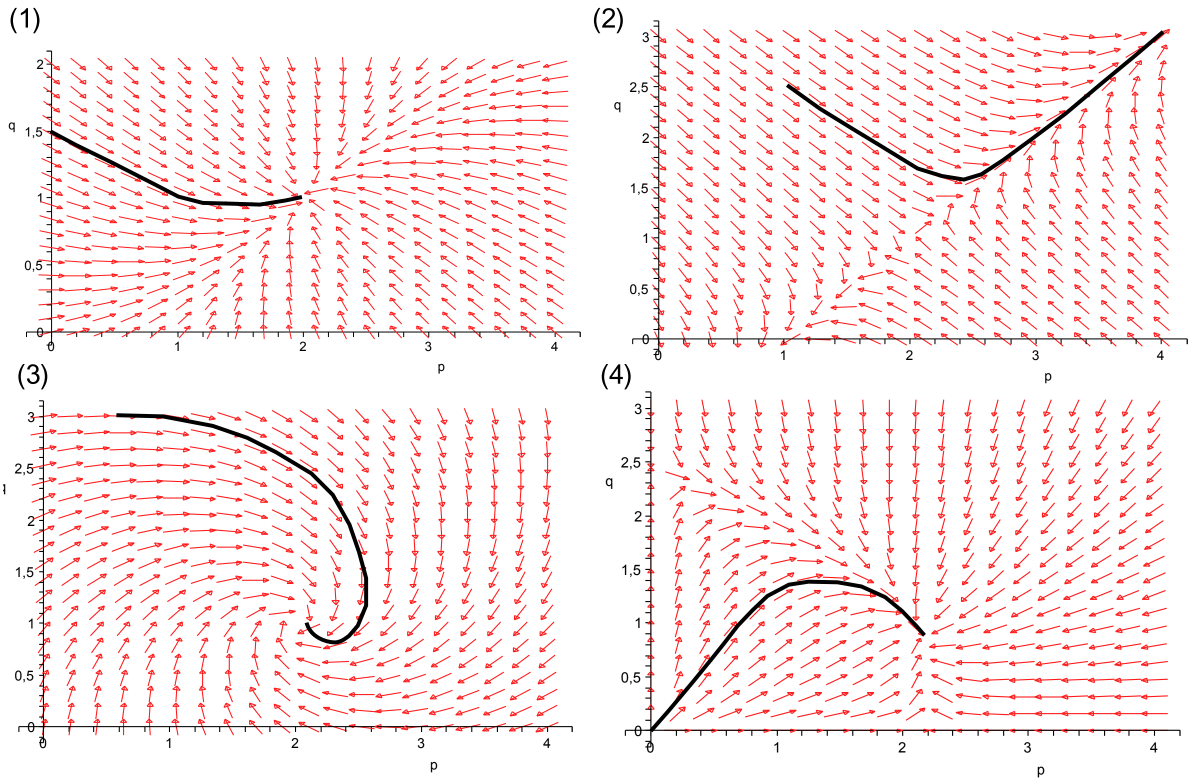


Figure 4: Sketches of the solution examples trajectories.

(H*) Exercise 4: Exponential Function for Matrices

Similar to scalars, the exponential function can be extended to matrices. It is defined for a matrix $A \in \mathbb{R}^{N \times N}$ as

$$exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

and can be used to analytically solve systems of ordinary differential equations.
Consider the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(a) Compute A^2, A^3, A^4 and A^5 and derive a general formula for A^{2k} and A^{2k+1}, $k \in \mathbb{N}$.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}^{2\times 2}$, $f(t) := exp(At)$. Show that

$$f(t) = I \cdot \cos(t) + A \cdot \sin(t)$$

where I is the identity matrix.
Hint: Use the results from (a) and consider the series representation of the trigonometric functions.

Solution:

(a) Computing the respective matrix products delivers:

$$\begin{aligned} A^2 &= -I \\ A^3 &= -A \\ A^4 &= I \\ A^5 &= A \end{aligned}$$

We can–via induction–conclude:

$$\begin{aligned} A^{2k} &= (-1)^k \cdot I \\ A^{2k+1} &= (-1)^k \cdot A \end{aligned}$$

(b) The trigonometric functions can be written as series as follows (see any good analysis text book):

$$\begin{aligned} \sin(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \\ \cos(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} \end{aligned}$$

We start with writing the function $f(t)$ according to the exponential series representation; here, we split the sum into odd and even summands:

$$f(t) = exp(A \cdot t) = \sum_{k=0}^{\infty} \frac{1}{k!}(A \cdot t)^k = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} \cdot t^{2k+1}$$

From (a), we can insert the expressions for A^{2k} and A^{2k+1}:

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} \cdot t^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot I \cdot t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot A \cdot t^{2k+1} \\ &= I \cdot \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot t^{2k} + A \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot t^{2k+1} \end{aligned}$$

As the matrices I and A are not part of the summands anymore, we immediately see that the remaining two sums exactly correspond to the sin- and cos-definitions from above:

$$\begin{aligned} f(t) &= I \cdot \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \cdot t^{2k} + A \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k \cdot t^{2k+1} \\ &= I \cdot \cos(t) + A \cdot \sin(t) \end{aligned}$$