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Worksheet 6
Problems

Ordinary Differential Equations: Numerical Methods

(H) Exercise 1: Convergence of the Euler Method

Consider the ODE

$$\frac{dy(t)}{dt} = Ay(t) + b \tag{1}$$

with $A \in \mathbb{R}^{N\times N}$, $y(t) : \mathbb{R}^N \to \mathbb{R}^N$ and $b \in \mathbb{R}^N$ (this could for example be the linear system arising from the two-species model). The explicit Euler method applied to this equation reads:

$$y^{(n+1)} = y^{(n)} + \tau(Ay^{(n)} + b) \tag{2}$$

with time step τ and $y^{(n)} := y(n \cdot \tau)$.

(a) Show the following statement: if the Euler method converges towards a vector y^*, then y^* must be a critical point of the ODE.

(b) Under which conditions does the Euler discretization from above (equation (2)) converge towards a critical point y^*?

(I) Exercise 2: Roadrunner vs Coyote

The Roadrunner is escaping Coyote. It therefore follows a (one-dimensional) line and changes its position $x(t)$ over time according to its current velocity $v(t) \geq 0$. The velocity arises from the acceleration/deceleration $a(t)$ of the Roadrunner. The motion of the Roadrunner can thus be modeled by a system of two ODEs:

$$\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= a(t) \end{aligned} \tag{3}$$

with initial conditions $x(0) = x_0$, $v(0) = v_0$.

(a) The Roadrunner decelerates as soon as it is fast and far away from its enemy. We thus assume that $a(t) = -v(t)$. Discretize the system of ODEs from equation (3) by the method

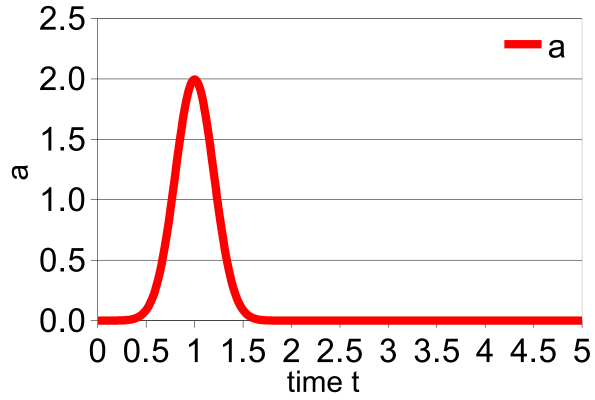

Figure 1: Acceleration $a(t)$ plotted over time t.

of Heun assuming the given deceleration and a time step $\tau = \frac{1}{10}$. For which time steps τ do we observe the correct physical behavior for this particular deceleration term (hint: consider the eigenvalues and eigenvectors of the linear time-stepping scheme)?

(b) In the Roadrunner-Coyote cartoons, the roadrunner performs very fast accelerations and decelerations. A respective time-dependent acceleration $a(t)$ to model the escape from the Coyote is sketched in Figure 1. Use the acceleration curve from Figure 1 to compute the approximate solution at $v(t = 1.0)$ with the explicit and the implicit Euler method and a time step $\tau = 1.0$. What do you observe in both cases? Which methodology do you suggest to improve the respective schemes considering both accuracy and computational efficiency?

(H) Exercise 3: Analysis of Single-Step Methods

Consider the ODE from last time

$$\frac{d^2y}{dt^2} = -y$$

and its transform into a first-order system of ODEs

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} \tag{4}$$

(a) Formulate the discrete update rule for the first-order system of equation (4) when applying the following single-step methods and using a time step τ:

- explicit Euler method
- implicit Euler method
- trapezoidal rule (Crank-Nicolson)

Write down the respective update scheme in matrix-vector form as

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = A_{method} \cdot \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} \tag{5}$$

where A_{method} denotes the method- and time step-dependent matrix for each of the single-step methods from above and $y^n := y(n \cdot \tau)$. What can you say about the long-time behavior of the system, that is for (y_0^n, y_1^n) when $n \to \infty$?

(b) Write a python script and check your analytical findings. You may consider solving the ODE from equation (4) for the initial values $y(0) = 0, dy(0)/dt = 1$.

(H*) Exercise 4: Analysis of a System of ODEs

The following system of ordinary differential equations is given:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_1(t) + \tfrac{1}{2} y_2(t) \\ \frac{dy_2(t)}{dt} &= \tfrac{1}{2} y_2(t), \end{aligned} \tag{6}$$

together with initial conditions $y_1(0) = 1, y_2(0) = 1$.

(a) Compute the critical points of the problem and the eigenvalues and eigenvectors of the matrix $A \in \mathbb{R}^{2\times 2}$ of the system $\frac{dy}{dt} = A \cdot y$. Draw the $y_1 - y_2$-direction field on the interval $[-1; 1] \times [-1; 1]$. Use the direction field to determine whether the critical points are stable, unstable or saddle points.

(b) Formulate the Crank-Nicolson (identical to second-order Adams-Moulton) method for the ODE from equation(6) using a time step τ. Compute the explicit form of the arising update scheme for $y_1(t + \tau)$, $y_2(t + \tau)$ (your computations need to be clear, and each step needs to be comprehensible).

Remark: you may use the following formula to invert 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{7}$$

Solution (explicit form of Crank-Nicolson):

$$y^{(n+1)} = \begin{pmatrix} \dfrac{1 + \frac{\tau}{2}}{1 - \frac{\tau}{2}} & \dfrac{\frac{\tau}{2}}{\left(1 - \frac{\tau}{2}\right)\left(1 - \frac{\tau}{4}\right)} \\ 0 & \dfrac{1 + \frac{\tau}{4}}{1 - \frac{\tau}{4}} \end{pmatrix} y^{(n)} \tag{8}$$

(c) Consider the eigenvalues of the Crank-Nicolson matrix in equation (8). For which time steps τ do you expect instabilities? Explain your decision by a short computation.