Technische Universität München
Lehrstuhl für Informatik V
Dr. T. Neckel
M.Sc. D. Jarema

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Worksheet 6
Sample Solutions

Ordinary Differential Equations: Numerical Methods

(H) Exercise 1: Convergence of the Euler Method

Consider the ODE

$$\frac{dy(t)}{dt} = Ay(t) + b \tag{1}$$

with $A \in \mathbb{R}^{N\times N}$, $y(t) : \mathbb{R}^N \to \mathbb{R}^N$ and $b \in \mathbb{R}^N$ (this could for example be the linear system arising from the two-species model). The explicit Euler method applied to this equation reads:

$$y^{(n+1)} = y^{(n)} + \tau(Ay^{(n)} + b) \tag{2}$$

with time step τ and $y^{(n)} := y(n \cdot \tau)$.

(a) Show the following statement: if the Euler method converges towards a vector y^*, then y^* must be a critical point of the ODE.

(b) Under which conditions does the Euler discretization from above (equation (2)) converge towards a critical point y^*?

Solution:

(a) Let the Euler method converge towards y^*, that is $y^{(n)} \stackrel{n\to\infty}{\longrightarrow} y^*$. This implies that for the vector y^*, it needs to hold:

$$y^* = y^* + \tau(Ay^* + b) \quad \Leftrightarrow \quad 0 = \tau(Ay^* + b) \quad \Leftrightarrow \quad 0 = Ay^* + b$$

The latter equation exactly corresponds to the condition for a critical point of the ODE from equation (1).

(b) We first re-write the Euler update rule from equation (2) in a more compact form:

$$y^{(n+1)} = y^{(n)} + \tau\left(Ay^{(n)} + b\right) = (I + \tau A)y^{(n)} + \tau b =: My^{(n)} + c, \tag{3}$$

with $M := I + \tau A \in \mathbb{R}^{N\times N}$ and $c := \tau b \in \mathbb{R}^N$. Then, we can look at the sequence $y^{(1)}$, $y^{(2)}$, ..., $y^{(n+1)}$:

$$\begin{aligned}
y^{(1)} &= My^{(0)} + c \\
y^{(2)} &= My^{(1)} + c = M(My^{(0)} + c) + c = M^2 y^{(0)} + Mc + c \\
y^{(3)} &= My^{(2)} + c = M(M^2 y^{(0)} + Mc + c) + c = M^3 y^{(0)} + M^2 + Mc + c \\
&\vdots \\
y^{(n+1)} &= M^{n+1} y^{(0)} + M^n c + \ldots + Mc + c = M^{n+1} y^{(0)} + \sum_{k=0}^{n} M^k c
\end{aligned} \tag{4}$$

We further assume that we can write the initial vector $y^{(0)}$ and c as linear combinations of the eigenvectors v_i of M:

$$\begin{aligned}
y^{(0)} &= \sum_i \alpha_i v_i \\
c &= \sum_i \beta_i v_i
\end{aligned} \tag{5}$$

with coefficients α_i, β_i. Inserting this into equation (4) yields:

$$\begin{aligned}
y^{(n+1)} &= \sum_i \alpha_i \lambda_i^{n+1} v_i + \sum_{k=0}^{n} \sum_i \beta_i \lambda_i^k v_i \\
&= \sum_i \left(\alpha_i \lambda_i^{n+1} v_i + \beta_i v_i \sum_{k=0}^{n} \lambda_i^k \right) \\
&= \sum_i \left(\alpha_i \lambda_i^{n+1} + \beta_i \sum_{k=0}^{n} \lambda_i^k \right) v_i
\end{aligned} \tag{6}$$

Let's consider the contributions of each eigenvector; therefore, we consider the cases $\lambda_i = 1$ and $\lambda_i \neq 1$:

$$\begin{aligned}
\lambda_i \neq 1: &\quad \alpha_i \lambda_i^{n+1} + \beta_i \frac{1 - \lambda_i^{n+1}}{1 - \lambda_i} \\
\lambda_i = 1: &\quad \alpha_i + (n+1)\beta_i
\end{aligned} \tag{7}$$

For the first case in equation (7), we made use of the compact writing for geometric sums. In order to test for convergence towards a critical point, we can analyze the long-time behavior of each eigenvector contribution:

- Assume that the matrix A has an eigenvalue $\mu_i > 0$ and a respective eigenvector w_i. Then, we have:

  $$Mw_i = (I + \tau A) w_i = w_i + \tau \mu_i w_i = (1 + \tau \mu_i) w_i$$

  $\lambda_i = 1 + \tau \mu_i$ is thus an eigenvalue of M and $v_i = w_i$ is the corresponding eigenvector. Since $\mu_i > 0$, the eigenvalue λ_i must be bigger than one, $\lambda_i = 1 + \tau \mu_i > 1$. As a consequence, the respective factor from equation (7) tends towards plus/ minus infinity or remains zero:

$$\alpha_i(1+\tau\mu_i)^{n+1}+\beta_i\frac{1-(1+\tau\mu_i)^{n+1}}{-\tau\mu_i}$$

$$= \left(\alpha_i+\frac{\beta_i}{\tau\mu_i}\right)(1+\tau\mu_i)^{n+1}-\frac{\beta_i}{\tau\mu_i} \quad \xrightarrow{n\to\infty} \quad \begin{cases} \infty & \text{if} \quad \alpha_i+\frac{\beta_i}{\tau\mu_i}>0 \\ -\infty & \text{if} \quad \alpha_i+\frac{\beta_i}{\tau\mu_i}<0 \\ -\frac{\beta_i}{\tau\mu_i} & \text{if} \quad \alpha_i+\frac{\beta_i}{\tau\mu_i}=0 \end{cases} \tag{8}$$

- If we have an eigenvalue of A which fulfills $\mu_i < -2/\tau$, then it follows $\lambda_i = 1+\tau\mu_i < 1+\tau\cdot(-2/\tau) = -1$ for the corresponding eigenvalue of M. For the case that $\alpha_i + \frac{\beta_i}{\tau\mu_i} \neq 0$ in equation (8), no convergence is reached with the summand oscillating between $\pm\infty$.
- If all eigenvalues of A fulfill $-2/\tau < \mu_i < 0$, then all summands from equation (8) tend towards $-\beta_i/(\tau\mu_i)$. We thus obtain:

$$\begin{aligned} y^{(n+1)} &= \sum_i \left(\left(\alpha_i+\frac{\beta_i}{\tau\mu_i}\right)(1+\tau\mu_i)^{n+1}-\frac{\beta_i}{\tau\mu_i}\right)v_i \\ &\xrightarrow{n\to\infty} \sum_i \left(-\frac{\beta_i}{\tau\mu_i}\right)v_i \end{aligned} \tag{9}$$

From (a), we know that the converged solution is also a critical point of our ODE. We further see that the critical point only depends on the coefficients β_i, but not on α_i. This means that the critical point is solely defined via the vector $c = \tau \cdot b$, but it does not depend on the initial condition $y^{(0)}$. The initial condition which enters our update rule via $M^{n+1}y^{(0)}$ decays over time and tends towards zero.
The condition $-2/\tau < \mu_i < 0$ is typically interpreted as restriction for the time step: convergence towards a critical point can only be obtained if the time step τ is chosen small enough such that it satisfies:

$$-\frac{2}{\tau} < \mu_i \quad \Leftrightarrow \quad \tau < -\frac{2}{\mu_i}$$

Remark: so far we investigated a case of real eigenvalues $\mu \in \mathbb{R}$. For a general case we have to consider inequality

$$|1+\tau\mu_i| < 1 \tag{10}$$

From this inequality follows that

$$\tau < -2\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2}, \tag{11}$$

so that the numerical solution converges.

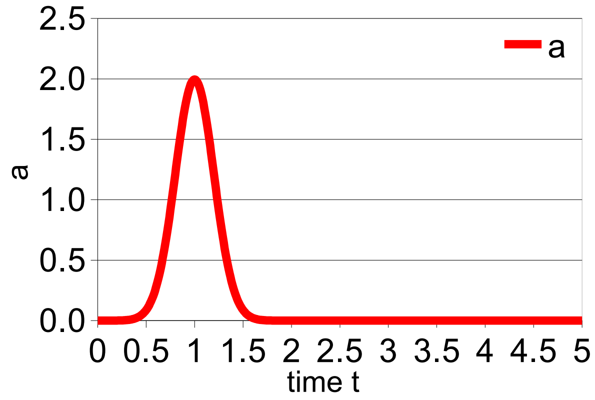

Figure 1: Acceleration $a(t)$ plotted over time t.

(I) Exercise 2: Roadrunner vs Coyote

The Roadrunner is escaping Coyote. It therefore follows a (one-dimensional) line and changes its position $x(t)$ over time according to its current velocity $v(t) \geq 0$. The velocity arises from the acceleration/deceleration $a(t)$ of the Roadrunner. The motion of the Roadrunner can thus be modeled by a system of two ODEs:

$$\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= a(t) \end{aligned} \tag{12}$$

with initial conditions $x(0) = x_0$, $v(0) = v_0$.

(a) The Roadrunner decelerates as soon as it is fast and far away from its enemy. We thus assume that $a(t) = -v(t)$. Discretize the system of ODEs from equation (12) by the method of Heun assuming the given deceleration and a time step $\tau = \frac{1}{10}$. For which time steps τ do we observe the correct physical behavior for this particular deceleration term (hint: consider the eigenvalues and eigenvectors of the linear time-stepping scheme)?

(b) In the Roadrunner-Coyote cartoons, the roadrunner performs very fast accelerations and decelerations. A respective time-dependent acceleration $a(t)$ to model the escape from the Coyote is sketched in Figure 1. Use the acceleration curve from Figure 1 to compute the approximate solution at $v(t = 1.0)$ with the explicit and the implicit Euler method and a time step $\tau = 1.0$. What do you observe in both cases? Which methodology do you suggest to improve the respective schemes considering both accuracy and computational efficiency?

Solution:

(a) For $a(t) = -v(t)$, we obtain a linear system of ODEs $dy/dt = Ay$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ v \end{pmatrix} \tag{13}$$

with $y = (x, v)^\top$ and matrix $A \in \mathbb{R}^{2\times 2}$. Applying the method of Heun yields:

$$\begin{aligned} y(t+\tau) &= y(t) + \tfrac{\tau}{2}\left(Ay(t) + A(y(t) + \tau Ay(t))\right) \\ &= \left(I + \tau A + \tfrac{\tau^2}{2}A^2\right) y(t) \end{aligned} \tag{14}$$

Computing A^2 yields

$$A^2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

We thus have:

$$y(t+\tau) = \begin{pmatrix} 1 + \frac{1}{10}0 + \frac{1}{200}0 & 0 + \frac{1}{10}1 + \frac{1}{200}(-1) \\ 0 + \frac{1}{10}0 + \frac{1}{200}0 & 1 + \frac{1}{10}(-1) + \frac{1}{200}1 \end{pmatrix} y(t) = \begin{pmatrix} 1 & \frac{19}{200} \\ 0 & \frac{181}{200} \end{pmatrix} y(t) \tag{16}$$

Considering the form of equation (16), we see that the update matrix is an upper triangular matrix. The eigenvalues are thus located on the diagonal: $\lambda_1 = 1 + \tau \cdot 0 + \frac{\tau^2}{2} \cdot 0 = 1$, $\lambda_2 = 1 - \tau + \frac{\tau^2}{2}$. The first eigenvalue corresponds to $(1, 0)^\top$ eigenvector and the second to $(1, -1)^\top$ eigenvector. Only the second eigenvector has the velocity component not equal zero and thus it determines the evolution of the velocity. Since the term $a(t) \leq 0$, the discrete system must have an eigenvalue $0 \leq \lambda_2 < 1$ to account for a continuous reduction in the velocity. Re-writing the eigenvalue λ_2 as $\lambda_2 = \frac{1}{2}(\tau - 1)^2 + \frac{1}{2}$ shows that the eigenvalue is always bigger than zero. Solving the equation

$$\lambda_2 \overset{!}{=} 1 \Leftrightarrow \tau = 0 \text{ or } \tau = 2 \tag{17}$$

yields that the time step always needs to be smaller than 2.

(b) The explicit Euler method reads $v(t+\tau) = v(t) + \tau a(t)$. Since $a(0) = 0$, we obtain $v(t+\tau) = 0$ for $\tau = 1.0$. The implicit Euler method reads $v(t+\tau) = v(t) + \tau a(t+\tau)$. Since $a(t+\tau) \approx 2.0$, we analogously obtain $v(t+\tau) = 0 + 1.0 \cdot 2.0 = 2.0$. The explicit Euler thus strongly underestimates the velocity whereas the implicit Euler overestimates the velocity (since it chooses the maximum acceleration over the whole time interval). In order to accurately resolve the very fast accelerations, a very fine time step is required at the strong peak. After the strong peak, the acceleration will only change slightly and bigger time steps are expected to be sufficient. This may also affect stability ($\rightarrow$ bigger time steps) and will strongly increase computational cost ($\rightarrow$ fine time steps). A possible solution is to adaptively choose the time step according to the magnitude of the acceleration and current velocity. This yields a compromise between accuracy and efficiency.

(H) Exercise 3: Analysis of Single-Step Methods

Consider the ODE from last time

$$\frac{d^2y}{dt^2} = -y$$

and its transform into a first-order system of ODEs

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} \tag{18}$$

(a) Formulate the discrete update rule for the first-order system of equation (18) when applying the following single-step methods and using a time step τ:

- explicit Euler method
- implicit Euler method
- trapezoidal rule (Crank-Nicolson)

Write down the respective update scheme in matrix-vector form as

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = A_{method} \cdot \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} \tag{19}$$

where A_{method} denotes the method- and time step-dependent matrix for each of the single-step methods from above and $y^n := y(n \cdot \tau)$. What can you say about the long-time behavior of the system, that is for (y_0^n, y_1^n) when $n \to \infty$?

(b) Write a python script and check your analytical findings. You may consider solving the ODE from equation (18) for the initial values $y(0) = 0, dy(0)/dt = 1$.

Solution:

(a) Let's start with the explicit Euler:

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} + \tau \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} \tag{20}$$

The implicit Euler update reads:

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} + \tau \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} \Leftrightarrow \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -\tau \\ \tau & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} \tag{21}$$

Note that for this particular scenario, an explicit update rule can be obtained for the implicit Euler with

$$A_{impl.\ Euler} := \begin{pmatrix} 1 & -\tau \\ \tau & 1 \end{pmatrix}^{-1}.$$

In general, the matrix may not be known a priori, may be too complex or too huge. A simple and computationally cheap inversion as in this example is therefore often not possible. Then, a linear system of equations needs to be solved for each time step:

$$\begin{pmatrix} 1 & -\tau \\ \tau & 1 \end{pmatrix} \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix}$$

Solving this system delivers the solution $(y_0^{n+1}, y_1^{n+1})^\top$ of the next time step.
A similar arguing holds for the trapezoidal rule where both solutions of time step n and $n+1$ are used to discretize the right hand side:

$$\begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} + \frac{\tau}{2} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix} + \frac{\tau}{2} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} \quad (22)$$

$$\Leftrightarrow \begin{pmatrix} 1 & -\frac{\tau}{2} \\ \frac{\tau}{2} & 1 \end{pmatrix} \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\tau}{2} \\ -\frac{\tau}{2} & 1 \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\tau}{2} \\ \frac{\tau}{2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \frac{\tau}{2} \\ -\frac{\tau}{2} & 1 \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} y_0^{n+1} \\ y_1^{n+1} \end{pmatrix} = \frac{1}{1+\frac{\tau^2}{4}} \begin{pmatrix} 1 - \frac{\tau^2}{4} & \tau \\ -\tau & 1 - \frac{\tau^2}{4} \end{pmatrix} \begin{pmatrix} y_0^n \\ y_1^n \end{pmatrix}$$

In order to investigate the long-time behavior, we can—similar to the previous investigations of discrete populations—analyze the eigenvalues of the update matrices

$$\begin{aligned} A_{expl.\ Euler} &:= \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \\ A_{impl.\ Euler} &:= \frac{1}{1+\tau^2} \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \\ A_{trapez.} &:= \frac{1}{1+\frac{\tau^2}{4}} \begin{pmatrix} 1 - \frac{\tau^2}{4} & \tau \\ -\tau & 1 - \frac{\tau^2}{4} \end{pmatrix} \end{aligned} \quad (23)$$

We obtain

- for the explicit Euler method: $\lambda_{1,2} = 1 \pm \tau i$
  The magnitude of both eigenvalues is bigger than one. This means that the solution will increase over time. The amplitude of the sine-waves of the analytical solution will hence grow bigger and bigger over time.

- for the implicit Euler method: $\lambda_{1,2} = \dfrac{1}{1 \pm \tau i}$
  These eigenvalues are exactly the inverse values of the eigenvalues from the explicit Euler method. Their magnitude is consequently < 1. This implies that the magnitude of the sine-waves will decay over time.
- for the trapezoidal rule: $\lambda_{1,2} = \dfrac{4 - \tau^2 \pm 4\tau i}{\tau^2 + 4}$
  Computing the magnitude of the eigenvalues delivers $\|\lambda_{1,2}\| = 1$. From this, we conclude that the magnitude of the solution is conserved over time. The amplitude of the sine-waves is thus expected to be conserved.

(b) See ws6_ex3.py

(H*) Exercise 4: Analysis of a System of ODEs

The following system of ordinary differential equations is given:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_1(t) + \tfrac{1}{2} y_2(t) \\ \frac{dy_2(t)}{dt} &= \tfrac{1}{2} y_2(t), \end{aligned} \tag{24}$$

together with initial conditions $y_1(0) = 1$, $y_2(0) = 1$.

(a) Compute the critical points of the problem and the eigenvalues and eigenvectors of the matrix $A \in \mathbb{R}^{2\times 2}$ of the system $\frac{dy}{dt} = A \cdot y$. Draw the $y_1 - y_2$-direction field on the interval $[-1;1] \times [-1;1]$. Use the direction field to determine whether the critical points are stable, unstable or saddle points.

(b) Formulate the Crank-Nicolson (identical to second-order Adams-Moulton) method for the ODE from equation(24) using a time step τ. Compute the explicit form of the arising update scheme for $y_1(t+\tau)$, $y_2(t+\tau)$ (your computations need to be clear, and each step needs to be comprehensible).

Remark: you may use the following formula to invert 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{25}$$

Solution (explicit form of Crank-Nicolson):

$$y^{(n+1)} = \begin{pmatrix} \dfrac{1 + \frac{\tau}{2}}{1 - \frac{\tau}{2}} & \dfrac{\frac{\tau}{2}}{\left(1 - \frac{\tau}{2}\right)\left(1 - \frac{\tau}{4}\right)} \\ 0 & \dfrac{1 + \frac{\tau}{4}}{1 - \frac{\tau}{4}} \end{pmatrix} y^{(n)} \tag{26}$$

(c) Consider the eigenvalues of the Crank-Nicolson matrix in equation (26). For which time steps τ do you expect instabilities? Explain your decision by a short computation.

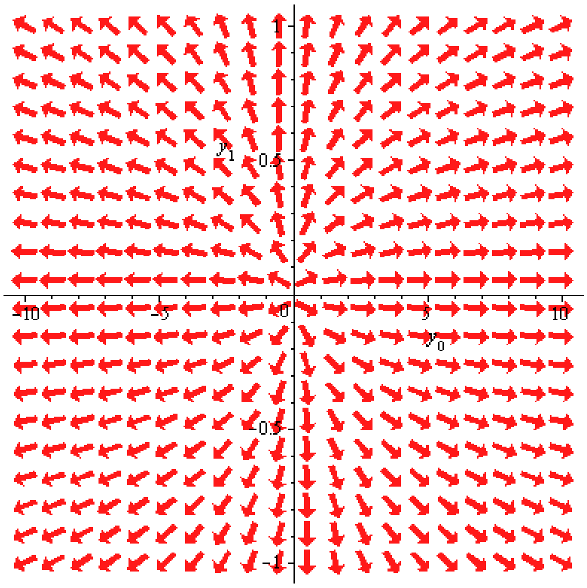

Figure 2: Direction field.

Solution:

(a) We first rewrite the system of ODEs in matrix-vector-form:

$$\frac{d\vec{y}}{dt} = \underbrace{\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}}_{A:=} \cdot \vec{y} \tag{27}$$

In order to find the critical points, we just need to solve the problem $A \cdot \vec{y} \overset{!}{=} \vec{0}$. The only solution is given by $y_1(t) = 0$, $y_2(t) = 0$.

Since the matrix is an upper triangular matrix, the eigenvalues are given by the entries on the diagonal: $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$. The eigenvectors evolve at $\vec{v}_1 = c \cdot (1, -1)^\top$, $\vec{v}_2 = c \cdot (1, 0)^\top$, $c \in \mathbb{R}$.

The direction field is given in Figure 2. All arrows tend away from the critical point. We thus have a single node as critical point which in the current case represents an unstable equilibrium point.

(b) The Crank-Nicolson method reads:

$$y^{(n+1)} = y^{(n)} + \frac{\tau}{2}\left(f(t^{(n)}, y^{(n)}) + f(t^{(n+1)}, y^{(n+1)})\right) \tag{28}$$

In the current example, we obtain:

$$y^{(n+1)} = y^{(n)} + \frac{\tau}{2}\left(\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} y^{(n)} + \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} y^{(n+1)}\right) \tag{29}$$

We first bring all contributions invoking $y^{(n+1)}$ to the left side:

$$\begin{pmatrix} 1-\frac{\tau}{2} & -\frac{\tau}{4} \\ 0 & 1-\frac{\tau}{4} \end{pmatrix} y^{(n+1)} = \begin{pmatrix} 1+\frac{\tau}{2} & \frac{\tau}{4} \\ 0 & 1+\frac{\tau}{4} \end{pmatrix} y^{(n)} \tag{30}$$

Inverting the matrix on the left side yields:

$$y^{(n+1)} = \frac{1}{(1-\frac{\tau}{4})(1-\frac{\tau}{2})} \begin{pmatrix} 1-\frac{\tau}{4} & \frac{\tau}{4} \\ 0 & 1-\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} 1+\frac{\tau}{2} & \frac{\tau}{4} \\ 0 & 1+\frac{\tau}{4} \end{pmatrix} y^{(n)} \tag{31}$$

This can be simplified to the given explicit update rule:

$$y^{(n+1)} = \begin{pmatrix} \frac{1+\frac{\tau}{2}}{1-\frac{\tau}{2}} & \frac{\frac{\tau}{2}}{(1-\frac{\tau}{2})(1-\frac{\tau}{4})} \\ 0 & \frac{1+\frac{\tau}{4}}{1-\frac{\tau}{4}} \end{pmatrix} y^{(n)} \tag{32}$$

(c) Instabilities occur if an eigenvalue becomes smaller than zero. This yields oscillations in our solution and our solution part-wise becomes negative (unphysical). The eigenvalues for the update matrix are given by $\lambda_1 = \frac{1+\frac{\tau}{2}}{1-\frac{\tau}{2}}$ and $\lambda_2 = \frac{1+\frac{\tau}{4}}{1-\frac{\tau}{4}}$. The numerator is positive for all $\tau > 0$ whereas the denominator can become negative in both cases. For $\tau > 2.0$, the eigenvalue λ_1 becomes negative. Hence, τ must be chosen smaller than 2.0.