

Scientific Computing I

Module 6: Analytical and Numerical Solutions of the 1D Heat Equation

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Part I: Analytical Solution

The Heat Equation in 1D

Analytical Solutions

- Separation of Variables

- Ensuring the Initial Conditions – Fourier's Method

The Heat Equation in 1D

- remember the heat equation without heat sources:

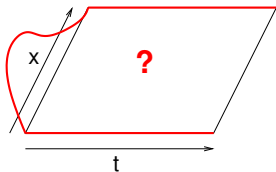
$$T_t = \kappa \Delta T$$

- we examine the 1D case, and set $\kappa = 1$ to get:

$$u_t = u_{xx} \quad \text{for } x \in (0, 1), t > 0$$

- using the following initial and boundary conditions:

$$\begin{aligned} u(x, 0) &= g(x), & x \in (0, 1) \\ u(0, t) &= u(1, t) = 0, & t > 0 \end{aligned}$$



Computing Analytical Solutions

We will, step by step, try to ...

1. find **any** solution of the PDE
(i.e., not necessarily **the** or **all** solutions)
2. that satisfy the boundary conditions
3. finally, satisfy initial conditions

Ansatz: **Separation of Variables**

- assumption:

$$u(x, t) = X(x) \cdot T(t)$$

- insert this assumption into the heat equation

Separation of Variables

- assumption:

$$u(x, t) = X(x) \cdot T(t)$$

- insert this assumption into the heat equation

$$\frac{\partial}{\partial t} (X(x) \cdot T(t)) = \frac{\partial^2}{\partial x^2} (X(x) \cdot T(t))$$

or

$$X(x) \cdot T_t(t) = T(t) \cdot X_{xx}(x)$$

- divide by $X(x)T(t)$, and get:

$$\frac{T_t(t)}{T(t)} = \frac{X_{xx}(x)}{X(x)}$$

- can be true if (and only if!) neither left- nor right-hand side depend on x or $t \Rightarrow$ both terms need to be constant!

Transforming the PDE into two ODEs

- separation of variables leads to:

$$\frac{T_t(t)}{T(t)} = \frac{X_{xx}(x)}{X(x)} = -\lambda.$$

- thus, we obtain two ODEs:

$$X_{xx}(x) + \lambda X(x) = 0 \quad (1)$$

$$T_t(t) + \lambda T(t) = 0. \quad (2)$$

- solve $X(x)$ -part:

$$X(x) = \sin(\sqrt{\lambda}x) \quad \text{or} \quad X(x) = \cos(\sqrt{\lambda}x),$$

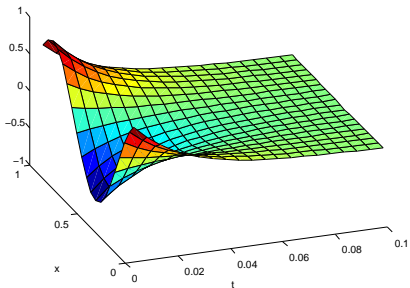
- solve $T(t)$ -part:

$$T(t) = e^{-\lambda t}.$$

Ensuring the Boundary Conditions

- We have computed general solutions

$$u(x, t) = \sin(\sqrt{\lambda}x) e^{-\lambda t} \text{ or } u(x, t) = \cos(\sqrt{\lambda}x) e^{-\lambda t}.$$



The boundary conditions of our example are

$$u(0, t) = u(1, t) = 0 \text{ for all } t.$$

- Thus, we can reduce possible solutions to

$$u_k(x, t) = \sin(k\pi x) e^{-(k\pi)^2 t}, \quad k = 1, 2, 3 \dots$$

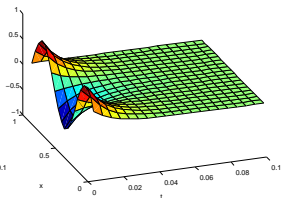
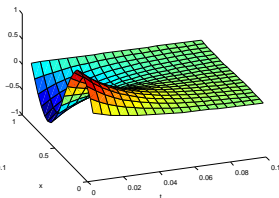
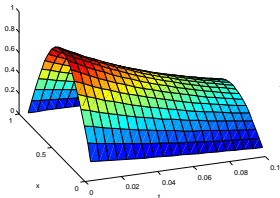
Ensuring the Initial Conditions – Fourier's Method

- The functions

$$u_k(x, t) = \sin(k\pi x)e^{-(k\pi)^2 t}, \quad k = 1, 2, \dots,$$

solve the 1D heat equation PDE with correct boundary values and for the initial conditions:

$$u_k(x, 0) = \sin(k\pi x), \quad x \in (0, 1).$$



- To ensure the initial conditions, we use the Fourier sine series:

$$g(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

Limits of Analytical Solutions

- For the 1D heat equation, we obtain an analytical solution

$$u(x, t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x).$$

- Solutions for the heat equation with heat sources?
Separation of variables yields

$$X(x) \cdot T_t(t) = T(t) \cdot X_{xx}(x) + f(x).$$

⇒ No analytical solution computable for arbitrary f .

- similar: unit square as computational domain?

⇒ **need for numerical methods!**

Part II: Numerical Solution

Numerical Solution 1 – An Explicit Scheme

- Discretisation

- Accuracy of the Explicit Scheme

- Stability of the Explicit Scheme

Numerical Solution 2 – An Implicit Scheme

- Implicit Time-Stepping

- Stability of the Implicit Scheme

Numerical Solution 1 – An Explicit Scheme

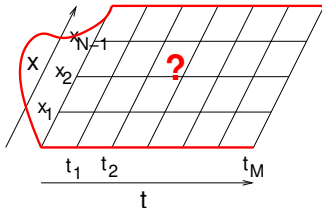
Discretisation similar to ODEs:

→ compute approximations

$$v_j^{(m)} \approx u(x_j, t_m)$$

at grid points x_j and time points t_m :

$$x_j := j \cdot h \quad t_m := m \cdot \tau,$$



approximate u_{xx} at (x_j, t_m) by $\frac{v_{j-1}^{(m)} - 2v_j^{(m)} + v_{j+1}^{(m)}}{h^2}$,

u_t at (x_j, t_m) by $\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau}$.

An Explicit Scheme

- This approximation results in the equations

$$\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau} = \frac{v_{j-1}^{(m)} - 2v_j^{(m)} + v_{j+1}^{(m)}}{h^2}. \quad (3)$$

for $j = 1, \dots, N - 1$ and $m \geq 0$.

- Add initial and boundary conditions:

$$\begin{aligned} v_0^{(m)} &= v_N^{(m)} = 0, & \text{for all } m \geq 0, \\ v_j^{(0)} &= g(x_j), & \text{for } j = 1, \dots, N - 1 \end{aligned}$$

- and obtain an explicit scheme:

$$v_j^{(m+1)} = v_j^{(m)} + \frac{\tau}{h^2} \left(v_{j-1}^{(m)} - 2v_j^{(m)} + v_{j+1}^{(m)} \right).$$

- We can, step by step, compute $v_j^{(m)}$ starting with $v_j^{(0)} = g(x_j)$.
- \Rightarrow Python Notebook demo

Accuracy of the Explicit Scheme

Observations:

- first order accurate in τ :

$$u_t(x, t) = \frac{u(x, t + \tau) - u(x, t)}{\tau} + \mathcal{O}(\tau)$$

- second order accurate in h :

$$u_{xx}(x, t) = \frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2} + \mathcal{O}(h^2)$$

Stability of the Explicit Scheme

- exact solution:

$$u_k(x, t) = e^{-(k\pi)^2 t} \sin(k\pi x)$$

- explicit scheme:

$$\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau} = \frac{v_{j-1}^{(m)} - 2v_j^{(m)} + v_{j+1}^{(m)}}{h^2}$$

Is there a stability condition for τ as observed in the ODE case?

- assumption: there are solutions of the form

$$v_j^{(m)} = (a_k)^m \sin(\pi k x_j), \quad \text{where } x_j := jh. \quad (4)$$

- compare with the exact solution $\Rightarrow (a_k)^m$ should decrease
 $\Rightarrow |a_k| < 1$ necessary

Stability of the Explicit Scheme (2)

- Inserting $v_j^{(m)} = (a_k)^m \sin(\pi k x_j)$ into

$$\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau} = \frac{v_{j-1}^{(m)} - 2v_j^{(m)} + v_{j+1}^{(m)}}{h^2}$$

leads to

$$a_k = 1 + \frac{\tau}{h^2} (2 \cos(\pi kh) - 2) = 1 - \frac{4\tau}{h^2} \sin^2 \left(\frac{\pi kh}{2} \right).$$

- for stability: $|a_k| < 1$ for all k if

$$\frac{4\tau}{h^2} < 2 \quad \text{or} \quad \tau < \frac{h^2}{2}.$$

for detailed computation, see tutorials!

Stability of the Explicit Scheme (3)

- solution for general initial conditions:

$$v_j^{(0)} = f_j = \sum_{k=1}^{n-1} c_k \sin(\pi k x_j),$$
$$v_j^{(m)} := \sum_{k=1}^{n-1} c_k (a_k)^m \sin(\pi k x_j)$$

- stability for $\tau < \frac{h^2}{2}$ because $|a_k| < 1$ for all k

Numerical Solution 2 – An Implicit Scheme

- apply implicit Euler:

$$\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau} = \frac{v_{j-1}^{(m+1)} - 2v_j^{(m+1)} + v_{j+1}^{(m+1)}}{h^2}$$

for $j = 1, \dots, n - 1$, and $m \geq 0$.

- boundary conditions:

$$v_0^{(m)} = v_N^{(m)} = 0, \quad \text{for all } m \geq 0,$$

- initial conditions:

$$v_j^{(0)} = g(x_j), \quad \text{for } j = 1, \dots, N - 1.$$

An Implicit Scheme

- implicit equations for $v_j^{(m+1)}$:

$$v_j^{(m+1)} - \frac{\tau}{h^2} \left(v_{j-1}^{(m+1)} - 2v_j^{(m+1)} + v_{j+1}^{(m+1)} \right) = v_j^{(m)}.$$

- With the ratio $r := \tau/h^2$, we can write it as

$$-rv_{j-1}^{(m+1)} + (1 + 2r)v_j^{(m+1)} - rv_{j+1}^{(m+1)} = v_j^{(m)}$$

for $j = 1, \dots, N - 1$, and $m \geq 0$.

- solution:

$$v^{(m+1)} = (I + rA)^{-1} v^{(m)},$$

where $A = \text{tridiag}(-1, 2, -1)$.

- Use a solver for linear system of equations to obtain $v_j^{(m+1)}$ in every step.
- \Rightarrow Python Notebook demo

Stability of the Implicit Scheme

- implicit scheme:

$$\frac{v_j^{(m+1)} - v_j^{(m)}}{\tau} = \frac{v_{j-1}^{(m+1)} - 2v_j^{(m+1)} + v_{j+1}^{(m+1)}}{h^2}$$

- again: insert assumed solutions

$$v_j^{(m)} = (a_k)^m \sin(\pi k x_j), \quad \text{where } x_j := jh$$

- into the implicit scheme, and obtain (see tutorials):

$$a_k = \left(1 + \frac{4\tau}{h^2} \sin^2 \left(\frac{\pi kh}{2} \right) \right)^{-1}.$$

- $0 < a_k < 1$ independent of k and h

⇒ unconditionally stable

Part III

Energy Considerations

Energy of the Analytic Solution

- $u(x, t)$ a solution of

$$\begin{aligned}(u_k)_t &= (u_k)_{xx}, & x \in (0, 1), t > 0 \\ u_k(0, t) = u_k(1, t) &= 0, & t > 0 \\ u_k(x, 0) &= g(x), & x \in (0, 1).\end{aligned}$$

- define an **“energy”** function of the solution (mathematical concept, not physical energy!):

$$E(t) := \int_0^1 u^2(x, t) dx$$

- for conservation of energy, analyse

$$E'(t) := \frac{d}{dt} \int_0^1 u^2(x, t) dx$$

Energy of the Analytic Solution (2)

$$\begin{aligned} E'(t) &= \int_0^1 \frac{\partial}{\partial t} u^2(x, t) dx = \int_0^1 2u(x, t)u_t(x, t) dx \\ &= 2 \int_0^1 u(x, t)u_{xx}(x, t) dx \\ &= 2 [u(x, t)u_x(x, t)]_0^1 - 2 \int_0^1 (u_x(x, t))^2 dx \\ &= -2 \int_0^1 (u_x(x, t))^2 dx \leq 0 \end{aligned}$$

Therefore:

- $E(t) \leq E(0)$ (energy never increases)
- compare to initial condition $u(x, 0) = g(x)$:

$$\int_0^1 u^2(x, t) dx = E(t) \leq E(0) = \int_0^1 g^2(x) dx$$

Corollaries

- assume: both $u_1(x, t)$ and $u_2(x, t)$ are solutions for initial conditions $g_1(x)$ and $g_2(x)$
- let $w(x, t) := u_1(x, t) - u_2(x, t)$, then

$$\begin{aligned}w_t(x, t) &= (u_1)_t(x, t) - (u_2)_t(x, t) \\ &= (u_1)_{xx}(x, t) - (u_2)_{xx}(x, t) = w_{xx}(x, t) \\ w(0, t) &= w(1, t) = 0 \\ w(x, 0) &= u_1(x, 0) - u_2(x, 0) = f_1(x) - f_2(x)\end{aligned}$$

- therefore, $w(x, t)$ is a solution of the heat equation for initial condition $g_w(x) = g_1(x) - g_2(x)$

Corollary 1 – Uniqueness

- if $g_1 = g_2$, then $g_w(x) = 0$
- energy is decreasing:

$$\begin{aligned}\int_0^1 (u_1 - u_2)^2(x, t) dx &= \int_0^1 w^2(x, t) dx \\ &\leq \int_0^1 (g_1 - g_2)^2(x) dx = 0\end{aligned}$$

- therefore:

$$\int_0^1 w^2(x, t) dx \leq 0 \quad \Leftrightarrow \quad w = 0 \quad \Leftrightarrow \quad u_1 = u_2.$$

- proof of uniqueness of the solution!

Corollary 2 – Stability

- now: $g_2 = g_1 + \epsilon$ (ϵ small), then

$$\begin{aligned}\int_0^1 w^2(x, t) dx &\leq \int_0^1 (g_1 - g_2)^2(x) dx \\ &= \int_0^1 \epsilon(x)^2 dx\end{aligned}$$

- therefore:
 - if ϵ is small, the difference between u_1 and u_2 also has got to be small, i.e.
 - small perturbations in the initial conditions lead to small perturbations in the solution.
- **stability estimate** (condition) for the analytical solution!

Energy of the Numerical Solution

- we introduce the “discrete energy”:

$$E^{(m)} := h \sum_{j=1}^{n-1} \left(v_j^{(m)} \right)^2.$$

- we would like to show that:

$$E^{(m+1)} \leq E^{(m)} \quad \text{for } m \geq 0.$$

- thus, we will compute $\Delta E^{(m)} := E^{(m+1)} - E^{(m)}$:

$$\Delta E^{(m)} = h \sum_{j=1}^{N-1} \left(\left(v_j^{(m+1)} \right)^2 - \left(v_j^{(m)} \right)^2 \right)$$

Energy in the Explicit Scheme

- lengthy computation (see separate worksheet)
leads to stability condition:

$$\left(E^{(m+1)} - E^{(m)}\right) \leq 0, \quad \text{or} \quad E^{(m+1)} \leq E^{(m)}$$

is **only correct, if**

$$\frac{\tau}{h^2} \leq \frac{1}{2} \quad \text{or} \quad \tau \leq \frac{h^2}{2}.$$

- otherwise:
increasing energy (physically incorrect), leads to large oscillations in the solution

Energy for the Implicit Scheme

- analyse discrete energy

$$E^{(m)} := h \sum_{j=1}^{N-1} \left(v_j^{(m)} \right)^2 = h \left(\mathbf{v}^{(m)} \right)^T \mathbf{v}^{(m)}.$$

- change of energy in each time step (use $\mathbf{v}^{(m+1)} = M\mathbf{v}^{(m)}$):

$$\begin{aligned} \Delta E^{(m)} &= h \left(\left(\mathbf{v}^{(m+1)} \right)^T \mathbf{v}^{(m+1)} - \left(\mathbf{v}^{(m)} \right)^T \mathbf{v}^{(m)} \right) \\ &= h \left(\left(M\mathbf{v}^{(m)} \right)^T M\mathbf{v}^{(m)} - \left(\mathbf{v}^{(m)} \right)^T \mathbf{v}^{(m)} \right) \\ &= h \left(\left(\mathbf{v}^{(m)} \right)^T M^T M\mathbf{v}^{(m)} - \left(\mathbf{v}^{(m)} \right)^T \mathbf{v}^{(m)} \right) \\ &= h \left(\mathbf{v}^{(m)} \right)^T \left(M^T M - I \right) \mathbf{v}^{(m)} \end{aligned}$$

Energy for the Implicit Scheme (2)

- energy for the implicit scheme:

$$\Delta E^{(m)} = h \left(v^{(m)} \right)^T \left(M^T M - I \right) v^{(m)}$$

with the iteration matrix $M = \left(I + \frac{\tau}{h^2} A \right)^{-1}$

- examine eigenvalues of matrix $M^T M - I$
- result:
 - all eigenvalues < 0
 - therefore $\Delta E^{(m)} \leq 0$
 - implicit scheme stable for any τ and h

for detailed computation, see separate worksheet!