Scientific Computing I

Module 3: Population Modelling – Continuous Models (Parts I and II)

Michael Bader
Lehrstuhl Informatik V

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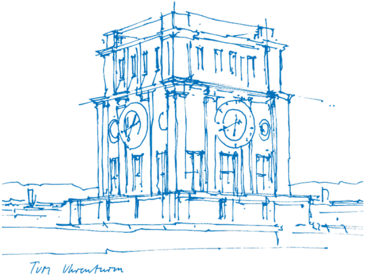


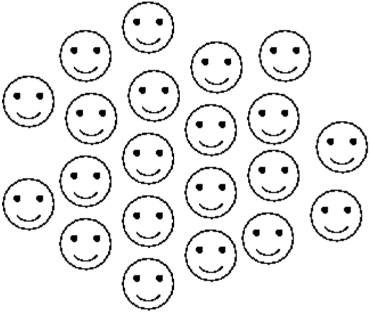
Part I: ODE Models

Part II: Discussion and Analysis of ODE Models

Part I

ODE Models

Discrete vs. Continuous Models



$$\frac{dp}{dt} = F(p, t, \dots)$$

$$p(t) = ?$$

discrete model:
$p(t) \in \mathbb{N}$ individuals

continuous model:
$p : \mathbb{R} \to \mathbb{R}, p(t) = ?$

Move to Continuous Models:

- easier(?) type of mathematical problem: differential equations, calculus
- analytical solutions available(?)

Model of Malthus (1798)

Only one species:

1. birth rate γ (number of births per time interval) proportional to size of population
2. death rate δ proportional to size of population
3. thus: constant growth (or decay) rate: $r = \gamma - \delta$

Modelling:

- constant growth rate

$$\frac{\Delta p}{\Delta t} = r \cdot p \qquad \text{or} \qquad \frac{dp}{dt} = r \cdot p$$

- growth within a time interval (cmp. Taylor series)

$$p(t + \Delta t) = p(t) + \Delta p(t) = p(t) + r \cdot p(t) \cdot \Delta t$$

Model of Malthus – Differential Equation

- Move to infinitesimally small time steps – note: $dp = dp(dt)$:

$$\lim_{dt \to 0} \frac{dp}{dt} = r \cdot p$$

- Written as an ordinary differential equation:

$$\dot{p}(t) = r \cdot p(t)$$

- Requires initial condition (population at start):

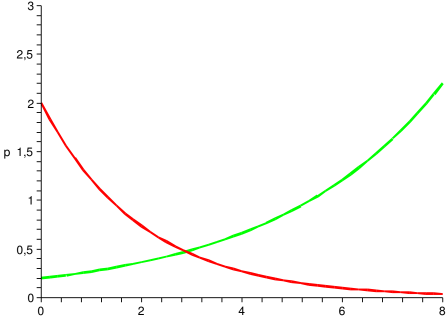
$$p(0) = p_0$$

- Analytical solution:

$$p(t) = p_0 \, e^{r\,t}$$

Model of Malthus – Solutions

The model of Malthus describes exponential growth or decay of a population:



Model of Verhulst (19th century)

Objective:

- model populations that approach saturation value

Assumptions:

- growth/death term depends on population size; assume linear dependency:

$$g(t) = g_0 - g_1 \cdot p(t) \qquad d(t) = d_0 + d_1 \cdot p(t)$$

- leads to differential equation:

$$\dot{p}(t) = g(t) - d(t) = \underbrace{(g_0 - d_0)}_{=:\alpha} - \underbrace{(g_1 + d_1)}_{=:\beta} p(t)$$

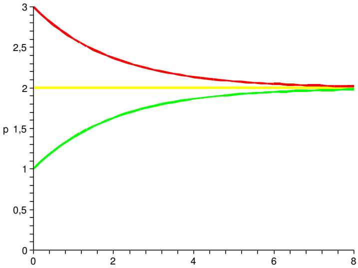
Model of Verhulst – Saturation

- solve initial value problem:

$$\dot{p}(t) = \alpha - \beta p(t), \qquad p(0) = p_0$$

- solution:

$$p(t) = p_\infty + e^{-\beta t}\left(p_0 - p_\infty\right), \qquad p_\infty = \frac{\alpha}{\beta}$$



Model of Verhulst – Logistic Growth

- saturation model does no longer model exponential growth
- idea: let growth/death **rate** decrease linearly with size of population
- but keep growth/death rate proportional to population size
- leads to differential equation:

$$\dot{p}(t) = (\alpha - \beta p(t))\, p(t)$$

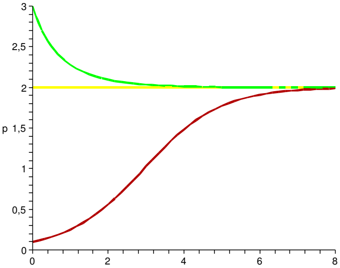
Logistic Growth

- other formulation

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

- solution:

$$p(t) = \frac{\beta}{(1 - e^{-\alpha t}) + \frac{\beta}{p_0} e^{-\alpha t}}$$

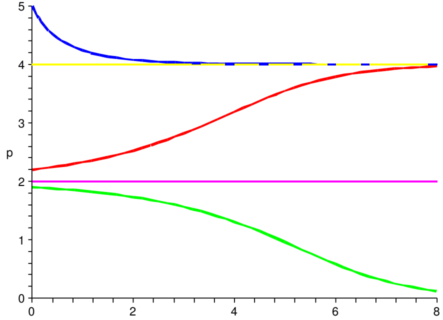


Logistic Growth with Threshold

- extended version of Verhulst's model:

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right)\left(1 - \frac{p(t)}{\delta}\right) p(t)$$

- solutions ($\beta = 2, \delta = 4$):



Example – The Passenger Pigeon

- beginning of the 19th century, estimated population in North America: four billion
- hunting diminished its number below a critical threshold (late 1880s)
- the last passenger pigeon died on Sep 1, 1914 (Cincinnati Zoo)

(photography taken at the Natural History Museum, London)

Part II

Discussion and Analysis of ODE Models

Analysis of ODE Models

Why Analyse a Given Model?

- analytical solutions difficult to compute
- properties of the solution not obvious:
  - shape of solutions?
  - possible steady state?
  - critical points? (species on edge of extinction?)

Methods to Improve Modeling?

- use analysis results to
  - predict failure of the model
  - tune parameters to model a specific situation

Analysing the Slope of a Solution

Example: Model of Malthus

$$\dot{p}(t) = \alpha p(t)$$

- for a physically reasonable solution: $p(t) > 0$
- α decides slope of solution:
  - $\alpha > 0$: growing population (accelerated growth)
  - $\alpha < 0$: receding population (decelerated reduction)

Equilibria and Critical Points

Example: Model of Verhulst (saturation)

$$\dot{p}(t) = \alpha - \beta p(t)$$

- equilibrium: $\dot{p}(t) = 0$
- only, if $p(t) = \frac{\alpha}{\beta}$

Example: Logistic Growth

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

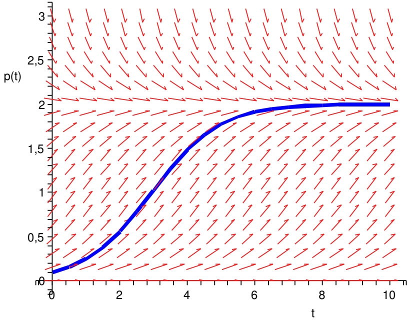
- constant solution, if $p(t) = \beta$ or $p(t) = 0$

Direction Field

plot derivatives vs. time and size of population:

Example: Logistic Growth

$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) p(t)$$

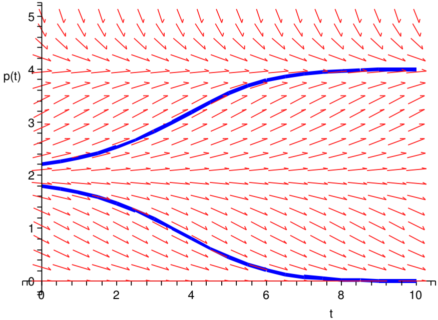


- $p = \beta$ reached for nearly all initial conditions
  $\rightarrow$ attractive/stable equilibrium/critical point
- $p = 0$ not reached for any other initial conditions
  $\rightarrow$ repulsive/unstable equilibrium/critical point

Direction Field (2)

Example: Logistic Growth with Threshold

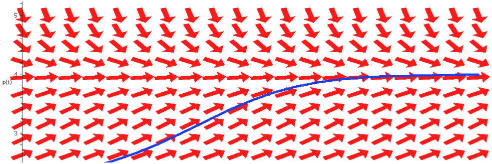
$$\dot{p}(t) = \alpha \left(1 - \frac{p(t)}{\beta}\right) \left(1 - \frac{p(t)}{\delta}\right) p(t)$$



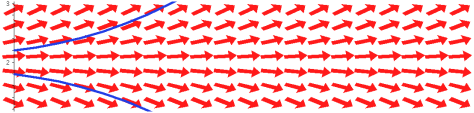
- stable critical points at $p = 0$ and $p = 4$
- unstable critical point at $p = 2$

Identifying Critical Points

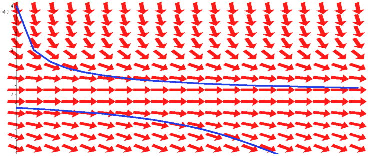
- attractive equilibrium:



- unstable equilibrium



- saddle point



Critical Points – Derivatives

Examine derivatives:

- critical point $p = \bar{p}$
- attractive equilibrium (asymptotically stable):

$$\begin{array}{lll} \dot{p} < 0 & \text{for} & p = \bar{p} + \epsilon \\ \dot{p} > 0 & \text{for} & p = \bar{p} - \epsilon \end{array}$$

- unstable equilibrium:

$$\begin{array}{lll} \dot{p} > 0 & \text{for} & p = \bar{p} + \epsilon \\ \dot{p} < 0 & \text{for} & p = \bar{p} - \epsilon \end{array}$$

- otherwise: saddle point