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Worksheet 11
Problems

Finite Element Methods

(H) Exercise 1: Weak Derivatives

Within the scope of finite element methods, the *weak formulation* of a partial differential equation is closely related to the notion of *weak (partial) derivatives*. In our example, let $C_0^\infty([-1,1])$ denote the space of all functions which are infinitely differentiable and are zero at the boundaries of the interval $[-1,1]$. For a function $v(x)$, we search for another function $w(x)$ which satisfies

$$\int_{-1}^{1} v(x)\varphi'(x)\mathrm{dx} = -\int_{-1}^{1} w(x)\varphi(x)\mathrm{dx} \tag{1}$$

for all $\varphi(x) \in C_0^\infty([-1,1])$. If both functions v, w are in the function space L^p, $w(x)$ is called *weak derivative* of $v(x)$. We won't go into details on L^p-function spaces at this stage since this is not a math course ;-) and this does not play a crucial role in this exercise[1].

(a) Show: if a function $v(x)$ is differentiable in the normal sense, i.e.

$$v'(x) := \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \tag{2}$$

exists, then the weak derivative is identical to $v'(x)$, $w(x) = v'(x)$.

(b) Consider the function definition for the absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}$,

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise.} \end{cases} \tag{3}$$

Determine the weak derivative of $v(x) := |x|$.

(H) Exercise 2: Test Functions

Consider the weak formulation for the one-dimensional Poisson equation with homogeneous Dirichlet conditions on the unit interval (that is $u(x=0) = u(x=1) = 0$):

$$\int_0^1 \nabla u(x) \cdot \nabla \varphi_i(x)\mathrm{dx} = \int_0^1 f(x)\varphi_i(x)\mathrm{dx} \quad \forall i \in \{1, ..., N\} \tag{4}$$

[1]For more information on these function spaces, see for example *Funktionalanalysis* by Dirk Werner.

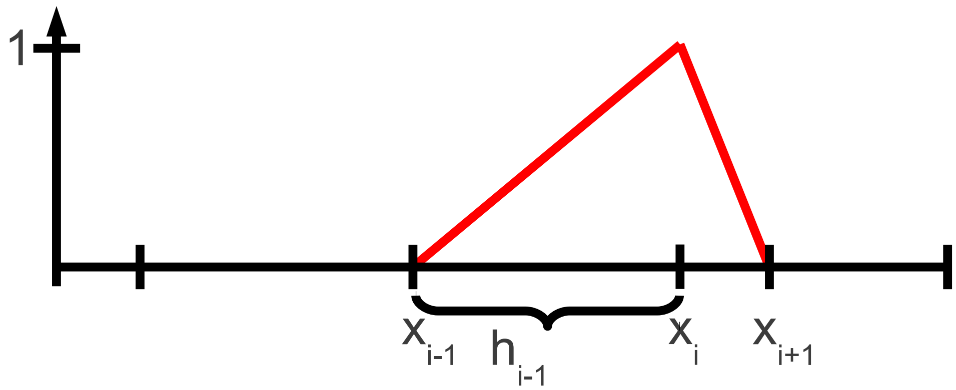

Figure 1: Hat function on an arbitrary one-dimensional grid.

The test functions $\varphi_i(x)$ shall span the respective test space. The discrete solution $u^h(x)$ can then be written as linear combination of the test functions, $u^h(x) := \sum_{j=1}^{N} a_j \varphi_j(x)$. This results in a linear system of equations for the coefficients a_j:

$$\sum_{j=1}^{N} a_j \int_0^1 \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \mathrm{dx} = \int_0^1 f(x) \varphi_i(x) \mathrm{dx} \quad \forall i \tag{5}$$

(a) Determine the entries of the matrix $A \in \mathbb{R}^{N \times N}$,

$$A_{ij} := \int_0^1 \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \mathrm{dx}, \tag{6}$$

for the following test functions:

1. $\varphi_i(x) := \sin(i\pi x)$, $i = 1, ..., N$; you may use the equality

$$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b).$$

2. the piecewise linear functions (hat functions) from the lecture. We therefore assume an *arbitrary* discretization of the unit interval with $N+2$ grid points (yielding N inner grid points) and a corresponding mesh size h_i, cf. Figure 1.

(b) Which properties does the matrix A in both scenarios have?

(I) Exercise 3: FEM for 1D Problem

We solve the one-dimensional partial differential equation

$$u_{xx}(x) + u(x) = f(x) \qquad x \in \Omega, \tag{7}$$

$$u(x) = 0 \qquad x \in \partial\Omega \tag{8}$$

on a domain Ω.

(a) Give the weak form of (7).

(b) We solve (7) using finite elements on a regular grid. Compute the element stiffness matrix for the interval displayed below with the linear nodal basis functions

$$\phi_1(x) = 1 - \frac{x}{h}, \tag{9}$$

$$\phi_2(x) = \frac{x}{h}. \tag{10}$$

where ϕ_i is the basis function associated to node i of the interval. Test and ansatz space are the same in our example (both given by $\phi_i, i = 1, 2$ on the interval).

(c) Use the results from (b) to assemble the global matrix for the following grid with five inner nodes: