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Worksheet 11
Sample Solutions

Finite Element Methods

(H) Exercise 1: Weak Derivatives

Within the scope of finite element methods, the *weak formulation* of a partial differential equation is closely related to the notion of *weak (partial) derivatives*. In our example, let $C_0^\infty([-1,1])$ denote the space of all functions which are infinitely differentiable and are zero at the boundaries of the interval $[-1,1]$. For a function $v(x)$, we search for another function $w(x)$ which satisfies

$$\int_{-1}^{1} v(x)\varphi'(x)\mathrm{dx} = -\int_{-1}^{1} w(x)\varphi(x)\mathrm{dx} \tag{1}$$

for all $\varphi(x) \in C_0^\infty([-1,1])$. If both functions v, w are in the function space L^p, $w(x)$ is called *weak derivative* of $v(x)$. We won't go into details on L^p-function spaces at this stage since this is not a math course ;-) and this does not play a crucial role in this exercise[1].

(a) Show: if a function $v(x)$ is differentiable in the normal sense, i.e.

$$v'(x) := \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \tag{2}$$

exists, then the weak derivative is identical to $v'(x)$, $w(x) = v'(x)$.

(b) Consider the function definition for the absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}$,

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise.} \end{cases} \tag{3}$$

Determine the weak derivative of $v(x) := |x|$.

Solution:

(a) All we need to show is that $w(x) = v'(x)$ fulfills equation (1) for every $\varphi \in C_0^\infty([-1,1])$. Let's choose a respective $\varphi(x)$ arbitrarily from this function space. Remember that our

[1] For more information on these function spaces, see for example *Funktionalanalysis* by Dirk Werner.

function $\varphi(x)$ thus automatically satisfies $\varphi(-1) = \varphi(1) = 0$. Inserting $v'(x)$ and $\varphi(x)$ into the right hand side of equation (1) and integrating by parts yield:

$$\begin{aligned} -\int_{-1}^{1} v'(x)\varphi(x)\mathrm{dx} &= \underbrace{-\left[v(x)\varphi(x)\right]_{-1}^{1}}_{=0,\text{ since } \varphi(-1)=\varphi(1)=0} + \int_{-1}^{1} v(x)\varphi'(x)\mathrm{dx} \\ &= \int_{-1}^{1} v(x)\varphi'(x)\mathrm{dx} \end{aligned} \tag{4}$$

We hence see that equation (1) is fulfilled.

(b) For an arbitrary function $\varphi(x)$ from the space $C_0^\infty([-1,1])$, we can write down the left hand side of equation (1) and transform it:

$$\begin{aligned} \int_{-1}^{1} |x|\varphi'(x)\mathrm{dx} &= \int_{-1}^{0} (-x)\varphi'(x)\mathrm{dx} + \int_{0}^{1} x\varphi'(x)\mathrm{dx} \\ &= -\int_{-1}^{0} x\varphi'(x)\mathrm{dx} + \int_{0}^{1} x\varphi'(x)\mathrm{dx} \\ &\overset{\text{integr. by parts}}{=} \underbrace{-\left[x\varphi(x)\right]_{-1}^{0}}_{=0} + \int_{-1}^{0} 1\cdot\varphi(x)\mathrm{dx} + \underbrace{\left[x\varphi(x)\right]_{0}^{1}}_{=0} - \int_{0}^{1} 1\cdot\varphi(x)\mathrm{dx} \\ &= \int_{-1}^{0} \varphi(x)\mathrm{dx} - \int_{0}^{1} \varphi(x)\mathrm{dx} \\ &= -\left(\int_{-1}^{0} -\varphi(x)\mathrm{dx} + \int_{0}^{1} \varphi(x)\mathrm{dx}\right) \end{aligned} \tag{5}$$

The last expression of equation (5) already looks very similar to the right hand side of equation (1). If we define $w(x)$ as the signum function,

$$w(x) := \begin{cases} -1 & x < 0 \\ 0 \quad \text{if} & x = 0 \\ 1 & x > 0, \end{cases} \tag{6}$$

we can rewrite equation (5) in a closed form over the whole interval $[-1,1]$:

$$\begin{aligned} \int_{-1}^{1} |x|\varphi'(x)\mathrm{dx} &= -\left(\int_{-1}^{0} -\varphi(x)\mathrm{dx} + \int_{0}^{1} \varphi(x)\mathrm{dx}\right) \\ &= -\int_{-1}^{1} w(x)\varphi(x)\mathrm{dx} \end{aligned} \tag{7}$$

The weak derivative of $|x|$ is thus given by the signum function.

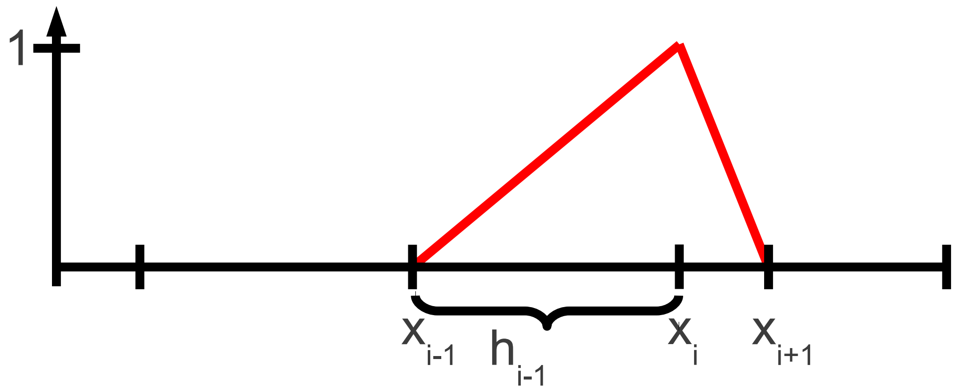


Figure 1: Hat function on an arbitrary one-dimensional grid.

(H) Exercise 2: Test Functions

Consider the weak formulation for the one-dimensional Poisson equation with homogeneous Dirichlet conditions on the unit interval (that is $u(x = 0) = u(x = 1) = 0$):

$$\int_0^1 \nabla u(x) \cdot \nabla \varphi_i(x) \mathrm{dx} = \int_0^1 f(x) \varphi_i(x) \mathrm{dx} \quad \forall i \in \{1, ..., N\} \tag{8}$$

The test functions $\varphi_i(x)$ shall span the respective test space. The discrete solution $u^h(x)$ can then be written as linear combination of the test functions, $u^h(x) := \sum_{j=1}^{N} a_j \varphi_j(x)$. This results in a linear system of equations for the coefficients a_j:

$$\sum_{j=1}^{N} a_j \int_0^1 \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \mathrm{dx} = \int_0^1 f(x) \varphi_i(x) \mathrm{dx} \quad \forall i \tag{9}$$

(a) Determine the entries of the matrix $A \in \mathbb{R}^{N \times N}$,

$$A_{ij} := \int_0^1 \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \mathrm{dx}, \tag{10}$$

for the following test functions:

1. $\varphi_i(x) := \sin(i\pi x)$, $i = 1, ..., N$; you may use the equality

$$\cos(a + b) + \cos(a - b) = 2\cos(a)\cos(b).$$

2. the piecewise linear functions (hat functions) from the lecture. We therefore assume an *arbitrary* discretization of the unit interval with $N + 2$ grid points (yielding N inner grid points) and a corresponding mesh size h_i, cf. Figure 1.

(b) Which properties does the matrix A in both scenarios have?

Solution:

(a) **Case 1**:

The derivative of $\varphi_i(x)$ is given by $\varphi'(x) = (i\pi)\cos(i\pi x)$. We can now consider the matrix entries A_{ij}:

$$\begin{aligned}
A_{ij} &= \int_0^1 \varphi_i'(x)\varphi_j'(x)\mathrm{dx} \\
&= ij\pi^2 \int_0^1 \cos(i\pi x)\cos(j\pi x)\mathrm{dx} \\
&\overset{\substack{\cos(a+b)+\cos(a-b)= \\ 2\cos(a)\cos(b)}}{=} \tfrac{1}{2}ij\pi^2 \int_0^1 \cos((i+j)\pi x) + \cos((i-j)\pi x)\mathrm{dx}
\end{aligned} \tag{11}$$

For $i \neq j$, we obtain:

$$\begin{aligned}
A_{ij} &= \tfrac{1}{2}ij\pi^2 \int_0^1 \cos((i+j)\pi x) + \cos((i-j)\pi x)\mathrm{dx} \\
&= \tfrac{1}{2}ij\pi^2 \left(\left[\tfrac{1}{(i+j)\pi}\sin((i+j)\pi x)\right]_0^1 + \left[\tfrac{1}{(i-j)\pi}\sin((i-j)\pi x)\right]_0^1 \right) \\
&= 0
\end{aligned} \tag{12}$$

For $i = j$, we obtain:

$$\begin{aligned}
A_{ii} &= \tfrac{1}{2}i^2\pi^2 \int_0^1 \cos(2i\pi x) + 1\mathrm{dx} \\
&= \tfrac{1}{2}i^2\pi^2 \left(\left[\tfrac{1}{2i\pi}\sin(2i\pi x)\right]_0^1 + [x]_0^1 \right) \\
&= \tfrac{1}{2}i^2\pi^2
\end{aligned} \tag{13}$$

A is hence a diagonal matrix.

Case 2:

We first define the hat function as described in Figure 1:

$$\varphi_i(x) := \begin{cases} \frac{1}{h_{i-1}}(x - x_{i-1}) & x \in [x_{i-1}, x_i] \\ \frac{1}{h_i}(x_{i+1} - x) \quad \text{if} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

This yields the gradient $\varphi_i'(x)$:

$$\varphi_i'(x) := \begin{cases} \frac{1}{h_{i-1}} & x \in [x_{i-1}, x_i] \\ -\frac{1}{h_i} \quad \text{if} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

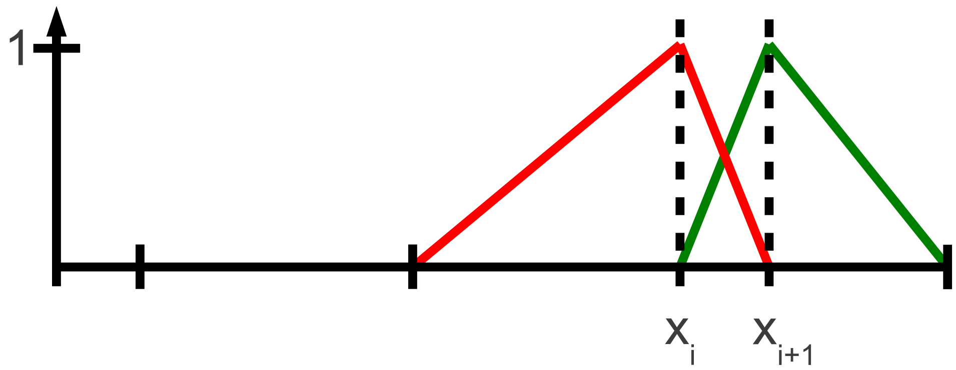


Figure 2: Overlap of two neighboring hat functions.

The gradient of the hat function is zero everywhere in $[0, x_{i-1}]$ and $[x_{i+1}, 1]$. For $j > i+1$, this means that

$$\begin{aligned} A_{ij} &= \int_0^1 \varphi_i'(x)\varphi_j'(x)\mathrm{dx} \\ &= \int_{x_{i-1}}^{x_{j+1}} \varphi_i'(x)\varphi_j'(x)\mathrm{dx} \\ &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)\cdot 0\mathrm{dx} + \int_{x_{j-1}}^{x_{j+1}} 0\cdot\varphi_j'(x)\mathrm{dx} \\ &= 0. \end{aligned} \tag{16}$$

The same holds for $j < i-1$. We hence only have to compute A_{ii},A_{ii+1} and A_{ii-1}. For A_{ii}, we obtain:

$$\begin{aligned} A_{ii} &= \int_0^1 \varphi_i'(x)^2\mathrm{dx} \\ &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)^2\mathrm{dx} \\ &= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_{i-1}}\right)^2\mathrm{dx} + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_i}\right)^2\mathrm{dx} \\ &= \frac{1}{h_{i-1}} + \frac{1}{h_i} \end{aligned} \tag{17}$$

The situation for A_{ii+1} is sketched in Figure 2. A non-vanishing integral only occurs on the interval $[x_i, x_{i+1}]$. We thus can compute the sub-diagonal entries A_{ii+1} as follows:

$$\begin{aligned} A_{ii+1} &= \int_0^1 \varphi_i'(x)\varphi_{i+1}'(x)\mathrm{d}x \\ &= \int_{x_i}^{x_{i+1}} \varphi_i'(x)\varphi_{i+1}'(x)\mathrm{d}x \\ &= \int_{x_i}^{x_{i+1}} \left(\tfrac{1}{h_i}\right)\left(-\tfrac{1}{h_i}\right)\mathrm{d}x = -\int_{x_i}^{x_{i+1}} \tfrac{1}{h_i^2}\mathrm{d}x \\ &= -\tfrac{1}{h_i} \end{aligned} \tag{18}$$

A_{ii-1} can be computed analogously:

$$\begin{aligned} A_{ii-1} &= \int_0^1 \varphi_i'(x)\varphi_{i-1}'(x)\mathrm{d}x \\ &= \int_{x_{i-1}}^{x_i} \varphi_i'(x)\varphi_{i-1}'(x)\mathrm{d}x \\ &= -\tfrac{1}{h_{i-1}} \end{aligned} \tag{19}$$

A single line of the matrix-vector form $A \cdot a$ (remember that $a = (a_1, ..., a_N)$ is the vector of coefficients to form $u^h = \sum_j a_j\varphi_j(x)$) can thus be written as

$$\sum_j A_{ij}a_j = A_{ii-1}a_{i-1} + A_{ii}a_i + A_{ii+1}a_{i+1} = -\frac{1}{h_{i-1}}a_{i-1} + \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)a_i - \frac{1}{h_i}a_{i+1}. \tag{20}$$

We observe that our Jacobi solver from Worksheet 8 may again work, since the magnitude of the diagonal element of our matrix rows is as big as the sum over the magnitudes of the sub-diagonal elements.

For equidistant grid spacing $h_i := h$, we obtain the well-known three-point stencil:

$$-\frac{1}{h_{i-1}}a_{i-1} + \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)a_i - \frac{1}{h_i}a_{i+1} = \frac{1}{h}(-a_{i-1} + 2a_i - a_{i+1}) \quad \Leftrightarrow \quad \frac{1}{h}[-1 \quad 2 \quad -1] \tag{21}$$

(b) The matrix A is obviously symmetric in both cases:

$$A_{ij} = \int_0^1 \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\mathrm{d}x = \int_0^1 \nabla\varphi_i(x) \cdot \nabla\varphi_j(x)\mathrm{d}x = A_{ji} \tag{22}$$

This is very nice: we have symmetry although we may have different grid spacing everywhere (with respect to the piecewise linear combination). The case 1 (sinus test functions) is less exciting, since we have a diagonal matrix. Hence, we consider the case of piecewise linear hat functions in more detail in the following. For this purpose, we check the matrix for positive definiteness and start from the original test space-independent definition of A:

$$
\begin{aligned}
y^\top A y &= \sum_i y_i \cdot \sum_j A_{ij} y_j = \sum_{i,j} y_i A_{ij} y_j \\
&= \sum_{i,j} y_i \left(\int_0^1 \varphi_i'(x) \varphi_j'(x) \mathrm{dx} \right) y_j \\
&= \int_0^1 \sum_{i,j} \varphi_i'(x) y_i \varphi_j'(x) y_j \mathrm{dx} \\
&= \int_0^1 \left(\sum_i \varphi_i'(x) y_i \right) \left(\sum_j \varphi_j'(x) y_j \right) \mathrm{dx} = \int_0^1 \underbrace{\left(\sum_i \varphi_i'(x) y_i \right)^2}_{\geq 0} \mathrm{dx} \geq 0
\end{aligned}
\tag{23}
$$

We thus observe that the matrix A is positive semi-definite *for any choice of $\varphi_i(x)$*. The function $\left(\sum_i \varphi_i'(x) y_i \right)^2$ is always greater or equal zero; the integral can thus only vanish if $\left(\sum_i \varphi_i'(x) y_i \right)^2$ is zero *on all sub-intervals* on $[0, 1]$. We can thus just look at each sub-interval $[x_i, x_{i+1}]$ and now re-insert the definition of the piecewise linear hat functions. On this interval, the expression can be further simplified:

$$
\begin{aligned}
& \int_{x_i}^{x_{i+1}} \left(\sum_i \varphi_i'(x) y_i \right)^2 \mathrm{dx} \\
=\; & \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_i} y_i + \frac{1}{h_i} y_{i+1} \right)^2 \mathrm{dx} \\
=\; & \frac{1}{h_i^2} \int_{x_i}^{x_{i+1}} (y_{i+1} - y_i)^2 \mathrm{dx} \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad y_i = y_{i+1}
\end{aligned}
\tag{24}
$$

The bilinear product $y^\top A y$ thus only vanishes if it holds $y_i = y_{i+1}$ for every i which implies that $y_1 = y_2 = ... = y_N = const$.

(I) Exercise 3: FEM for 1D Problem

We solve the one-dimensional partial differential equation

$$
u_{xx}(x) + u(x) = f(x) \qquad x \in \Omega, \tag{25}
$$

$$
u(x) = 0 \qquad x \in \partial\Omega \tag{26}
$$

on a domain Ω.

(a) Give the weak form of (25).

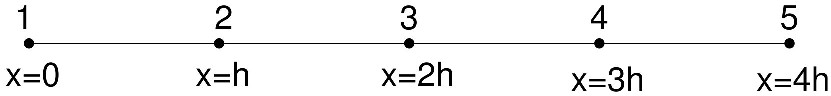
(b) We solve (25) using finite elements on a regular grid. Compute the element stiffness matrix for the interval displayed below with the linear nodal basis functions

$$\phi_1(x) = 1 - \frac{x}{h}, \quad (27)$$

$$\phi_2(x) = \frac{x}{h}. \quad (28)$$

where ϕ_i is the basis function associated to node i of the interval. Test and ansatz space are the same in our example (both given by $\phi_i, i = 1, 2$ on the interval).

(c) Use the results from (b) to assemble the global matrix for the following grid with five inner nodes:



Solution:

(a) Multiply with test function, integrate, and apply Green's formula:

$$\begin{aligned} \int_\Omega v(x)\left(u_{xx}(x) + u(x)\right) \mathrm{d}x &= \int_\Omega v(x) f(x) \mathrm{d}x, \quad \text{for all } v, \\ -\int_\Omega v_x(x) u_x(x) - v(x) u(x) \mathrm{d}x &= \int_\Omega v(x) f(x) \mathrm{d}x, \quad \text{for all } v. \end{aligned}$$

(b) Derivatives of the basis functions:

$$\begin{aligned} \phi_{1,x}(x) &= -\frac{1}{h}, \\ \phi_{2,x}(x) &= \frac{1}{h}. \end{aligned}$$

Compute matrix entries:

$$\begin{aligned} A_{1,1} &= -\int_0^h \left(\phi_{1,x}^2(x) - \phi_1^2(x)\right) \mathrm{d}x \\ &= -\int_0^h \left(\frac{1}{h^2} - \left(1 - \frac{x}{h}\right)^2\right) \mathrm{d}x \\ &= -\frac{1}{h} + \left[\frac{\left(1 - \frac{x}{h}\right)^3}{3} \cdot (-h)\right]_0^h \\ &= -\frac{1}{h} + \frac{h}{3}, \end{aligned}$$

$$\begin{aligned}
A_{1,2} &= -\int_0^h \left(\phi_{1,x}(x)\phi_{2,x}(x) - \phi_1(x)\phi_2(x)\right) \mathrm{d}x \\
&= -\int_0^h \left(-\frac{1}{h^2} - \left(1 - \frac{x}{h}\right)\frac{x}{h}\right) \mathrm{d}x \\
&= \frac{1}{h} + \left[\frac{x^2}{2h} - \frac{x^3}{3h^2}\right]_0^h \\
&= \frac{1}{h} + \frac{h}{6},
\end{aligned}$$

$$\begin{aligned}
A_{2,1} &= A_{1,2}, \\
A_{2,2} &= -\int_0^h \left(\phi_{2,x}^2(x) - \phi_2^2(x)\right) \mathrm{d}x \\
&= -\int_0^h \left(\frac{1}{h^2} - \frac{x^2}{h^2}\right) \mathrm{d}x \\
&= -\frac{1}{h} + \left[\frac{x^3}{3h^2}\right]_0^h \\
&= -\frac{1}{h} + \frac{h}{3}.
\end{aligned}$$

This results in the element matrix

$$\begin{pmatrix} -\frac{1}{h} + \frac{h}{3} & \frac{1}{h} + \frac{h}{6} \\ \frac{1}{h} + \frac{h}{6} & -\frac{1}{h} + \frac{h}{3} \end{pmatrix}.$$

(c) "Blow up" the element matrices and renumber to fit with the numbering of unknowns in the global two-element system: first (left) element:

$$\begin{pmatrix} -\frac{1}{h} + \frac{h}{3} & \frac{1}{h} + \frac{h}{6} & 0 & 0 & 0 \\ \frac{1}{h} + \frac{h}{6} & -\frac{1}{h} + \frac{h}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

second element:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{h} + \frac{h}{3} & \frac{1}{h} + \frac{h}{6} & 0 & 0 \\ 0 & \frac{1}{h} + \frac{h}{6} & -\frac{1}{h} + \frac{h}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

third element:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{h} + \frac{h}{3} & \frac{1}{h} + \frac{h}{6} & 0 \\ 0 & 0 & \frac{1}{h} + \frac{h}{6} & -\frac{1}{h} + \frac{h}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

fourth (right) element:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{h}+\frac{h}{3} & \frac{1}{h}+\frac{h}{6} \\ 0 & 0 & 0 & \frac{1}{h}+\frac{h}{6} & -\frac{1}{h}+\frac{h}{3} \end{pmatrix}$$

Add the four matrices:

$$\begin{pmatrix} -\frac{1}{h}+\frac{h}{3} & \frac{1}{h}+\frac{h}{6} & 0 & 0 & 0 \\ \frac{1}{h}+\frac{h}{6} & -\frac{2}{h}+\frac{2h}{3} & \frac{1}{h}+\frac{h}{6} & 0 & 0 \\ 0 & \frac{1}{h}+\frac{h}{6} & -\frac{2}{h}+\frac{2h}{3} & \frac{1}{h}+\frac{h}{6} & 0 \\ 0 & 0 & \frac{1}{h}+\frac{h}{6} & -\frac{2}{h}+\frac{2h}{3} & \frac{1}{h}+\frac{h}{6} \\ 0 & 0 & 0 & \frac{1}{h}+\frac{h}{6} & -\frac{1}{h}+\frac{h}{3} \end{pmatrix}.$$