Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
M.Sc. S. Seckler, M.Sc. D. Jarema, M.Sc. A. Bakhtiari

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Worksheet 12
Problems

Finite Element Methods

(H) Exercise 1: Hierarchical Basis Functions And Function Spaces

Until now, we assumed to search a function space $span(\varphi_i, i = 1, ..., N)$ for a suitable solution of a given partial differential equation. In the first part of this exercise, we want to consider how good the finite element approximation works if the solution belongs to a different space. Therefore, we consider the very simple equation

$$u(x) = -x^2 + 1 \tag{1}$$

which obviously has zero boundary values on the interval $[-1, 1]$. We hence already know the solution of the equation; based on this solution, we can investigate the approximation of finite elements.

(a) Derive the weak formulation of equation (1) for an arbitrary choice of test functions $\varphi_i(x)$.

(b) We introduce the hierarchical basis, cf. Figure 1: instead of defining a local piecewise linear approximation, we define hat functions which hierarchically approximate the space. Level 0 consists of only one grid point, level 1 adds 2 more grid points, and so forth. Set up the linear system of equations for the weak formulation from (a) using the basis function of level 0 and level 1. Solve the system via python script. Compare the solution to the one-point solution (which only involves the basis function on level 0).

(c) Consider the coefficients a_i of your numerical solution $u^h = \sum_i a_i \varphi_i(x)$ from (b). What behaviour do you expect for the coefficients when taking more levels of the hierarchical basis into account? How does the approximation change if we use the piecewise linear hat functions instead (and also use the same number of basis functions as before)?

(d) Next, the hierarchical basis is used to solve the Poisson problem. Compute the stiffness matrix $A \in \mathbb{R}^{3\times 3}$,

$$A_{ij} = \int_{-1}^{1} \nabla \varphi_i(x) \nabla \varphi_j(x) \mathrm{d}x \tag{2}$$

for the hierarchical basis functions on level 0 and level 1. What do you observe?

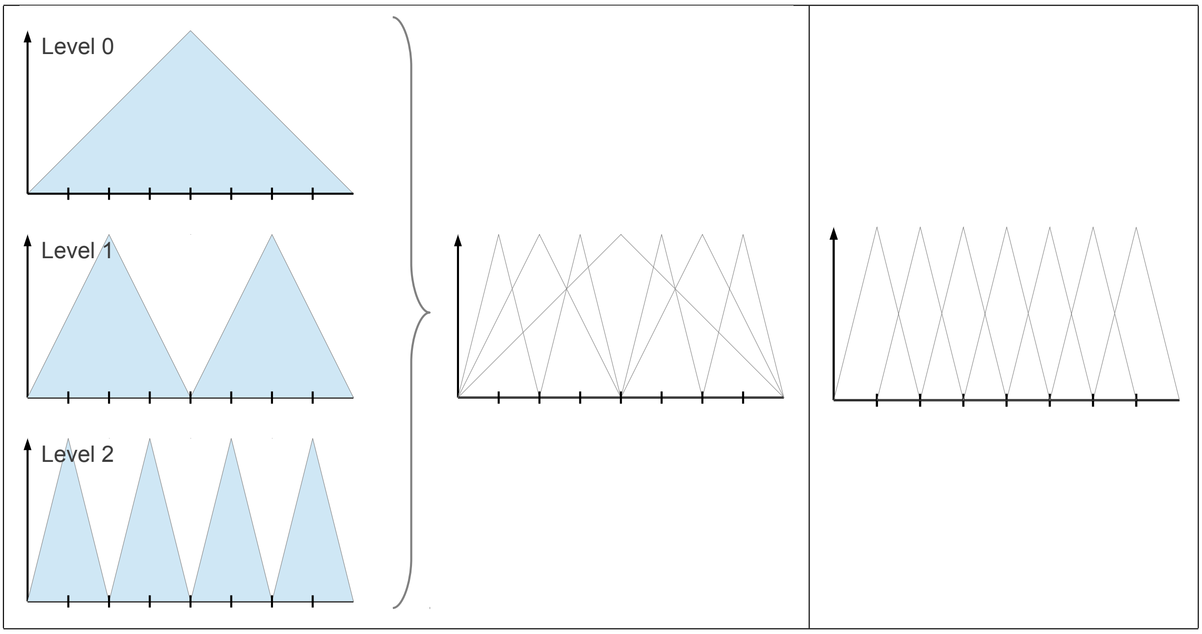

Figure 1: Left: composition of the hierarchical linear basis functions using levels 0,1 and 2. Right: piecewise linear hat functions.

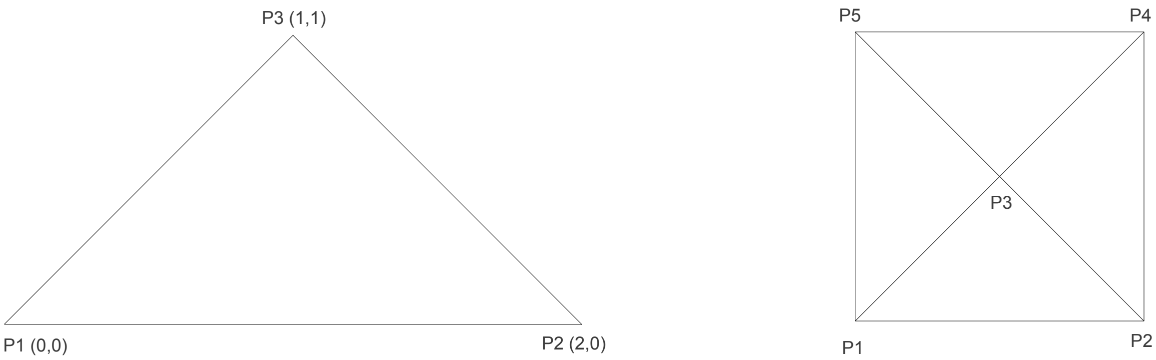

Figure 2: Left: reference triangle. Right: domain which is discretised by triangles of same shape (and size) as on the left.

(H) Exercise 2: Elementwise Assembling

A two-dimensional triangle, cf. Figure 2 on the left, is given as reference element. Piecewise linear functions are defined and shall be used as test functions; each function takes the value 1 at exactly one corner of the triangle and 0 at all other corners.

(a) Define the three functions $\varphi_1(x, y)$, $\varphi_2(x, y)$ and $\varphi_3(x, y)$ for a single triangular element and compute their gradients.

(b) Compute the elementwise expressions for the stiffness matrix $A_{ij} := \int_E \nabla\varphi_i \cdot \nabla\varphi_j$ where E denotes the area spanned by the triangle.

(c) Assemble the global stiffness matrix for the box domain shown in Figure 2 on the right using the elementwise derivations from (b).

(I) Exercise 3: Poisson Equation on a Triangular Grid

We solve the two-dimensional Poisson equation

$$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y) \in \Omega, \quad (3)$$
$$u(x,y) = 0 \text{ at } \partial\Omega \quad (4)$$

on a domain Ω.

(a) Give the weak form of (3).

(b) We solve (3) using finite elements on a triangular grid. Piecewise linear functions are used as test functions. Define these functions for the triangle displayed in the left Figure 3 and compute their gradients.

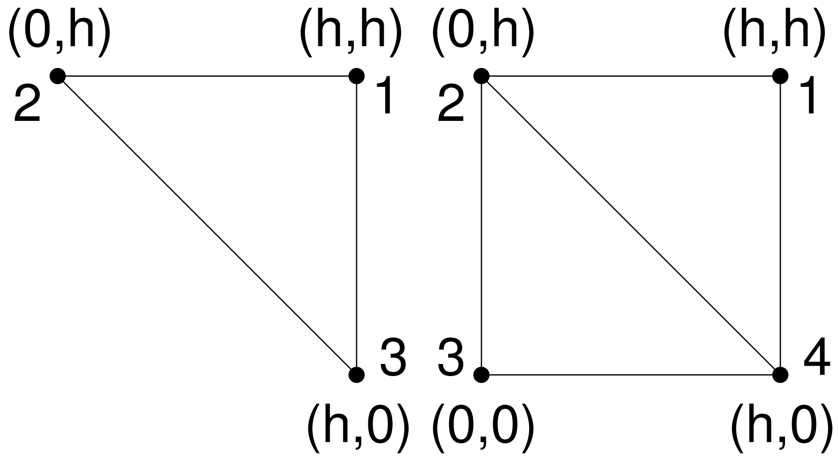

Figure 3: Left: Reference triangle. Right: Domain with two triangular elements.

(c) Compute the element stiffness matrix for the reference triangle. Test and ansatz space are the same in our example.

(d) Use the results from (c) to assemble the matrix for the grid consisting of two elements in the right Figure 3.

Hint: Consider only the given two elements without boundary conditions. Be careful with the numbering of nodes. The resulting two-element matrix should use the numbering given in the right Figure 3.

(e) Give a reason why you might want to assemble element stiffness matrices instead of a discretization stencil.