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Worksheet 12
Sample Solutions

Finite Element Methods

(H) Exercise 1: Hierarchical Basis Functions And Function Spaces

Until now, we assumed to search a function space $span(\varphi_i, i = 1, ..., N)$ for a suitable solution of a given partial differential equation. In the first part of this exercise, we want to consider how good the finite element approximation works if the solution belongs to a different space. Therefore, we consider the very simple equation

$$u(x) = -x^2 + 1 \tag{1}$$

which obviously has zero boundary values on the interval $[-1, 1]$. We hence already know the solution of the equation; based on this solution, we can investigate the approximation of finite elements.

(a) Derive the weak formulation of equation (1) for an arbitrary choice of test functions $\varphi_i(x)$.

(b) We introduce the hierarchical basis, cf. Figure 1: instead of defining a local piecewise linear approximation, we define hat functions which hierarchically approximate the space. Level 0 consists of only one grid point, level 1 adds 2 more grid points, and so forth. Set up the linear system of equations for the weak formulation from (a) using the basis function of level 0 and level 1. Solve the system via python script. Compare the solution to the one-point solution (which only involves the basis function on level 0).

(c) Consider the coefficients a_i of your numerical solution $u^h = \sum_i a_i \varphi_i(x)$ from (b). What behaviour do you expect for the coefficients when taking more levels of the hierarchical basis into account? How does the approximation change if we use the piecewise linear hat functions instead (and also use the same number of basis functions as before)?

(d) Next, the hierarchical basis is used to solve the Poisson problem. Compute the stiffness matrix $A \in \mathbb{R}^{3\times 3}$,

$$A_{ij} = \int_{-1}^{1} \nabla \varphi_i(x) \nabla \varphi_j(x) \mathrm{dx} \tag{2}$$

for the hierarchical basis functions on level 0 and level 1. What do you observe?

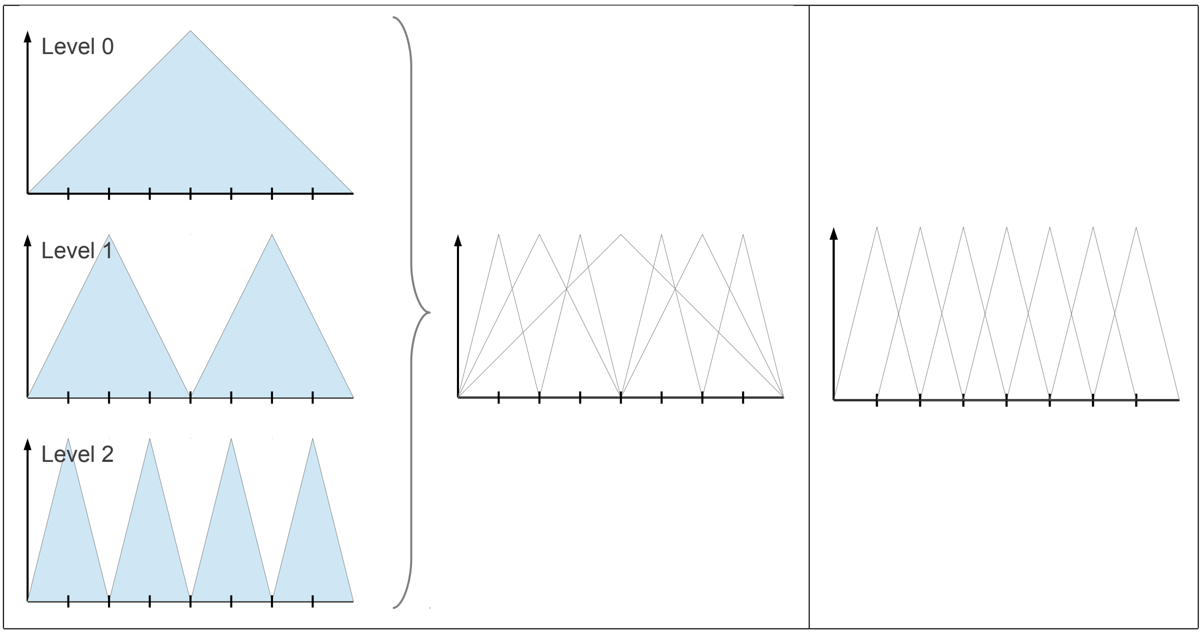

Figure 1: Left: composition of the hierarchical linear basis functions using levels 0,1 and 2. Right: piecewise linear hat functions.

Solution:

(a) We multiply equation (1) with a test function $\varphi_i(x)$ and integrate it over the interval $[-1, 1]$:

$$\int_{-1}^{1} u(x)\varphi_i(x)\mathrm{d}x = \int_{-1}^{1} (-x^2+1)\varphi_i(x)\mathrm{d}x \qquad \forall i \tag{3}$$

Inserting the approximation $u^h := \sum_j a_j\varphi_j(x)$ for $u(x)$ results in:

$$\sum_j a_j \int_{-1}^{1} \varphi_i(x)\varphi_j(x)\mathrm{d}x = \int_{-1}^{1} (-x^2+1)\varphi_i(x)\mathrm{d}x \qquad \forall i \tag{4}$$

(b) In order to setup and solve the finite element system from equation (4), we need to determine the mass matrix $M_{ij} := \int_{-1}^{1} \varphi_i(x)\varphi_j(x)\mathrm{d}x$ and the right hand side entries $r_i := \int_{-1}^{1}(-x^2+1)\varphi_i(x)\mathrm{d}x$. The arising linear system is thus given by

$$M \cdot a = r. \tag{5}$$

Since the hierarchical basis functions span over bigger intervals, the matrix M is not expected to be a tridiagonal matrix as in the case of using piecewise linear hat functions. However, considering Figure 1, we can see that only some basis functions "overlap" ; for example, the supports[1] of all basis functions that belong to the same level are disjoint. Besides, the left basis function on level 1 does not share any support with the two functions on the very right of level 2. We can hence still expect many zero entries in the

[1]The support of a function $f(x)$ is defined as the domain on which $f(x) \neq 0$.

matrix M. With respect to analytically computing M, let the basis functions of level 0 and level 1 be enumerated as follows:

- $\varphi_0(x)$: basis function on level 0
- $\varphi_1(x)$: left basis function on level 1
- $\varphi_2(x)$: right basis function on level 1

We want to consider only one entry of M here as example, that is the entry M_{02}. The respective basis functions are given by:

$$\varphi_0(x) := \begin{cases} x+1 & \text{if} \quad x \in [-1,0] \\ -x+1 & \text{if} \quad x \in [0,1] \end{cases} \quad , \quad \varphi_2(x) := \begin{cases} 2x & x \in [0,0.5] \\ -2x+2 & \text{if} \quad x \in [0.5,1] \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The computation reads:

$$\begin{aligned} M_{02} &= \int_{-1}^{1} \varphi_0(x)\varphi_2(x)\mathrm{dx} \\ &= \int_{0}^{1} \varphi_0(x)\varphi_2(x)\mathrm{dx} \\ &= \int_{0}^{0.5} \varphi_0(x)\varphi_2(x)\mathrm{dx} + \int_{0.5}^{1} \varphi_0(x)\varphi_2(x)\mathrm{dx} \\ &= \int_{0}^{0.5} (-x+1)2x\mathrm{dx} + \int_{0.5}^{1} (-x+1)(-2x+2)\mathrm{dx} \\ &= \left[-\tfrac{2}{3}x^3 + x^2\right]_0^{0.5} + \left[\tfrac{2}{3}x^3 - 2x^2 + 2x\right]_{0.5}^{1} \\ &= 0.25 \end{aligned} \tag{7}$$

Note that—due to the symmetry of the mass matrix M—it is sufficient to only compute the upper right triangular matrix of M.

For solving the system and integrating the hierarchical basis functions, see ws11_ex1.py.

(c) The coefficients of the three-point solution are given by

$$\begin{aligned} a_0 &= 1.035714285 \\ a_1 &= 0.2857142862 \\ a_2 &= 0.2857142862. \end{aligned} \tag{8}$$

The coefficients of the level 1-basis functions are smaller by a factor of four. Let's consider what happens when approximating a given parabola by the hierarchical basis, see Figure 2. On level 0, the coefficient of the hierarchical basis function is determined by the height of the parabola. On level 1, the coefficient arises from the height of the parabola without the contribution of the hierarchical basis function from level 0. This offset is illustrated as cyan line in Figure 2 on the left. On level 2, both contributions from level 0 and level 1 already define a certain height (red+green line). The coefficient

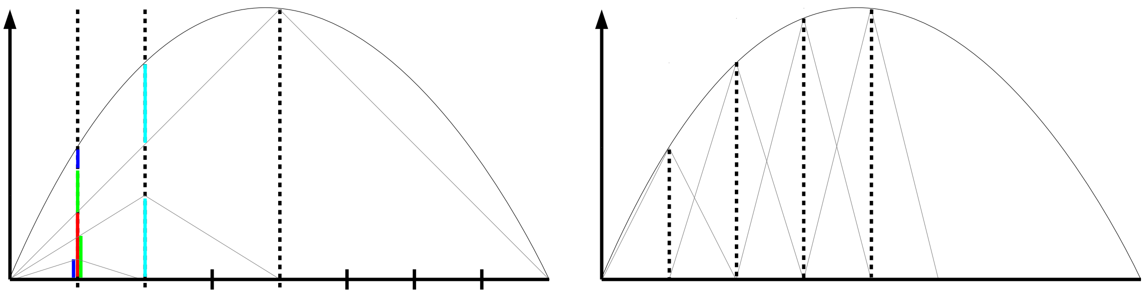

Figure 2: Illustration of the coefficients a_j for the hierarchical basis functions (left) and for the piecewise linear hat functions.

of the basis function of level 2 thus is given by the blue line = height of parabola − red line − green line. As one can observe, the respective coefficients become smaller with each additional level, since we already have a good approximation of the parabola via the basis functions of the coarser levels.

In Figure 2 on the right, the coefficients for the piecewise linear hat functions are shown (dashed lines). Each hat function only determines the function shape at its respective grid point, since it holds

$$\varphi_i(x_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{9}$$

where x_j denotes the center position for hat function $\varphi_j(x)$. Thus, the coefficients a_i are given by the value of the parabola at point x_i.
Still, both the piecewise linear hat functions and the hierarchical basis functions span the same function space, that is the space of piecewise linear functions. If we use the same grid points x_i, both finite element formulations hence must yield the same result.

(d) We first compute the diagonal element for the function $\varphi_0(x)$:

$$A_{00} = \int_{-1}^{1} (\nabla\varphi_0(x))^2 \mathrm{dx} = \int_{-1}^{0} 1^2 \mathrm{dx} + \int_{0}^{1} (-1)^2 \mathrm{dx} = \int_{-1}^{1} 1\mathrm{dx} = 2. \tag{10}$$

The other diagonal elements can be computed analogously; since the interval for the integration is reduced by a factor of 2 on the one hand on level 1 (compared to level 0) and the slope of the hat function increases by a factor of 2 on the other hand, the diagonal entries A_{11}, A_{22} have the value[2]

$$A_{11} = A_{22} = \frac{2 \cdot 2}{2} A_{00} = 2A_{00} = 4. \tag{11}$$

Now, let's consider one of the non-diagonal elements. We pick $\int \nabla\varphi_0 \cdot \nabla\varphi_2$:

$$\begin{aligned} \textstyle\int_{-1}^{1} \nabla\varphi_0(x) \cdot \nabla\varphi_2(x)\mathrm{dx} &= \textstyle\int_{0}^{1} \nabla\varphi_0(x) \cdot \nabla\varphi_2(x)\mathrm{dx} \\ &= \textstyle\int_{0}^{0.5} (-1) \cdot 2\mathrm{dx} + \int_{0.5}^{1} (-1) \cdot (-2)\mathrm{dx} = \cdots = 0 \end{aligned} \tag{12}$$

[2]The factor $2 \cdot 2/2$ arises as follows: factor $2 \cdot 2$ is due to the doubling of the slope of the hat function (factor 2 for each gradient $\varphi_i(x)$). The division by 2 comes from halvening the interval for the integration.

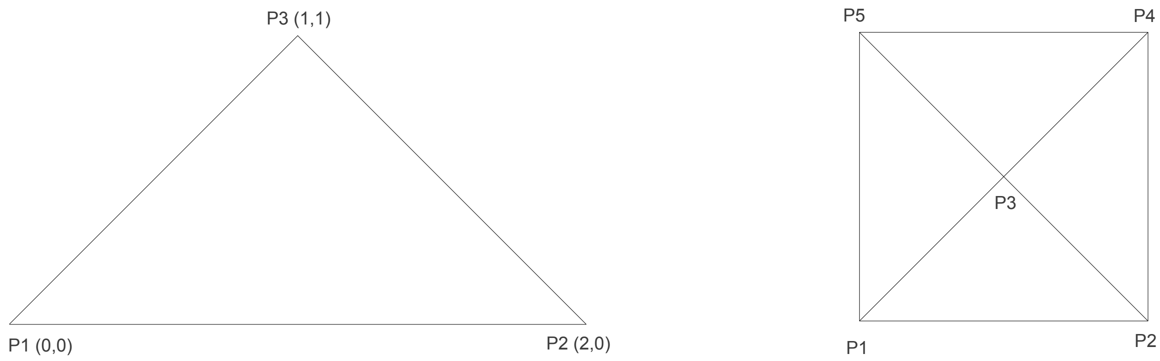

Figure 3: Left: reference triangle. Right: domain which is discretised by triangles of same shape (and size) as on the left.

Obtaining a value zero for this entry can be easily understood: on the whole integration interval $[0, 1]$, the basis function on level 0 has constant slope whereas the function $\varphi_2(x)$ has a slope ± 2. For this reason, the respective integrals over the constant functions cancel out. This holds for all other (non-diagonal) entries of the stiffness matrix whose basis functions share a certain overlap (basis functions which do not have any overlap yield vanishing entries in the stiffness matrix anyway). The stiffness matrix for the hierarchical basis functions is thus a diagonal matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \tag{13}$$

From this, we can see the big advantage of the hierarchical basis for the Poisson problem compared to the piecewise linear hat functions: the solution that we obtain in the end is in the same function space; solving the diagonal matrix-problem of the hierarchical basis representation, however, is much cheaper!

(H) Exercise 2: Elementwise Assembling

A two-dimensional triangle, cf. Figure 3 on the left, is given as reference element. Piecewise linear functions are defined and shall be used as test functions; each function takes the value 1 at exactly one corner of the triangle and 0 at all other corners.

(a) Define the three functions $\varphi_1(x, y)$, $\varphi_2(x, y)$ and $\varphi_3(x, y)$ for a single triangular element and compute their gradients.

(b) Compute the elementwise expressions for the stiffness matrix $A_{ij} := \int_E \nabla\varphi_i \cdot \nabla\varphi_j$ where E denotes the area spanned by the triangle.

(c) Assemble the global stiffness matrix for the box domain shown in Figure 3 on the right using the elementwise derivations from (b).

Solution:

(a) The functions are defined as follows on the given triangle:

$$\begin{aligned} \varphi_1(x,y) &:= 1 - \tfrac{1}{2}x - \tfrac{1}{2}y \\ \varphi_2(x,y) &:= \tfrac{1}{2}x - \tfrac{1}{2}y \\ \varphi_3(x,y) &:= y \end{aligned} \tag{14}$$

The gradients evolve at:

$$\begin{aligned} \frac{\partial\varphi_1(x,y)}{\partial x} &= -\frac{1}{2} & \frac{\partial\varphi_1(x,y)}{\partial y} &= -\frac{1}{2} \\ \frac{\partial\varphi_2(x,y)}{\partial x} &= \frac{1}{2} & \frac{\partial\varphi_2(x,y)}{\partial y} &= -\frac{1}{2} \\ \frac{\partial\varphi_3(x,y)}{\partial x} &= 0 & \frac{\partial\varphi_3(x,y)}{\partial y} &= 1 \end{aligned} \tag{15}$$

(b) Since all our derivatives are constant on the triangle E, it follows that $\nabla\varphi_i \cdot \nabla\varphi_j = C_{ij}$ where C_{ij} is a real constant. All entries A_{ij} can be written as follows:

$$\begin{aligned} M_{ij} &= \textstyle\int_E \nabla\varphi_i \cdot \nabla\varphi_j \\ &= \textstyle\int_E C_{ij} \\ &= C_{ij} \textstyle\int_E 1 \end{aligned} \tag{16}$$

It hence sufficient to compute the area of the triangle. This area can be determined easily to be $\int_E 1 = 0.5 * 2 * 1 = 1$. The coefficients C_{ij} can be determined to be:

$$\begin{aligned} C_{11} &= \tfrac{1}{2} \\ C_{12} &= 0 \\ C_{13} &= -\tfrac{1}{2} \\ C_{22} &= \tfrac{1}{2} \\ C_{23} &= -\tfrac{1}{2} \\ C_{33} &= 1 \end{aligned} \tag{17}$$

The stiffness matrix thus evolves from these coefficients inserted into equation (16):

$$A = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{18}$$

Remark: for non-constant gradients, the integration needs to be performed explicitly on the respective element. In the present case, for example, one could split the triangle into E_1 defined by the points $P1 = (0,0), (1,0), P3 = (1,1)$ and E_2 given by $(1,0), P2 = (2,0), P3 = (1,1)$. The integration on these sub-elements then can be written as:

$$\int_{E_1} 1 = \int_0^1 \int_0^x 1 dy\, dx, \quad \int_{E_2} 1 = \int_1^2 \int_0^{2-x} 1 dy\, dx \tag{19}$$

(c) For the demonstration we will assemble the stiffness matrix only for point $P3$ and $P1$, other matrix elements can be computed similarly.

We will introduce variables for the four elements shown in the figure:

- E_1: $P1$, $P2$, $P3$
- E_2: $P2$, $P4$, $P3$
- E_3: $P4$, $P5$, $P3$
- E_4: $P5$, $P1$, $P3$

For each point, we further define the respective test function $\varphi_i(x,y)$, $i = 1,...,5$. Next, the elementwise assembling of the stiffness matrix can be carried out for each entry $M_{3i} = \sum_k \int_{E_k} \nabla\varphi_i \cdot \nabla\varphi_3$.

$$\begin{aligned} A_{31} &= \sum_k \int_{E_k} \nabla\varphi_1 \cdot \nabla\varphi_3 \\ &= \int_{E_1} \nabla\varphi_1 \cdot \nabla\varphi_3 + \int_{E_4} \nabla\varphi_1 \cdot \nabla\varphi_3 \\ &= -\tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = -1 \end{aligned}$$

Let's step through this first computation. From line 1 to line 2, we determine the elements which are relevant for our computations. On elements E_2, E_3, the function $\varphi_1(x,y) = 0$ (by definition of the piecewise linear functions). It is thus sufficient to only integrate over E_1 and E_4. For each of these two elements, we can now consider the elementwise contribution for the stiffness matrix. Let $\tilde{\varphi}_i(x,y)$, $i = 1,2,3$, denote the *local* piecewise linear function on the reference triangle from part (b). Then, on element E_1, $\varphi_1(x,y)$ corresponds to $\tilde{\varphi}_1(x,y)$, and $\varphi_3(x,y)$ corresponds to $\tilde{\varphi}_3(x,y)$. Using the results from part (b), we see that the integral for element E_1 has been computed to be $-\frac{1}{2}$. Similarly, we can consider the second integral: on element E_4, $\varphi_1(x,y)$ corresponds to the local function $\tilde{\varphi}_2(x,y)$, and $\varphi_3(x,y)$ again corresponds to $\tilde{\varphi}_3(x,y)$. The integral is thus – according to part (b) – given by $-\frac{1}{2}$.

Due to symmetry, the other integral expressions for A_{ij}, $i \neq j$, must be identical. For the sake of completeness, all other integrals are listed and computed in the following:

$$\begin{aligned} A_{32} &= \sum_k \int_{E_k} \nabla\varphi_2 \cdot \nabla\varphi_3 \\ &= \int_{E_1} \nabla\varphi_2 \cdot \nabla\varphi_3 + \int_{E_2} \nabla\varphi_2 \cdot \nabla\varphi_3 \\ &= -\tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = -1 \\ A_{33} &= \sum_k \int_{E_k} \nabla\varphi_3 \cdot \nabla\varphi_3 \\ &= 1 + 1 + 1 + 1 = 4 \end{aligned}$$

$$
\begin{aligned}
A_{34} &= \sum_k \int_{E_k} \nabla\varphi_4 \cdot \nabla\varphi_3 \\
&= \int_{E_2} \nabla\varphi_4 \cdot \nabla\varphi_3 + \int_{E_3} \nabla\varphi_4 \cdot \nabla\varphi_3 \\
&= -\tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = -1 \\
A_{35} &= \sum_k \int_{E_k} \nabla\varphi_5 \cdot \nabla\varphi_3 \\
&= \int_{E_3} \nabla\varphi_5 \cdot \nabla\varphi_3 + \int_{E_4} \nabla\varphi_5 \cdot \nabla\varphi_3 \\
&= -\tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = -1
\end{aligned}
$$

We observe that we obtain the well-known stencil $[-1 \quad -1 \quad 4 \quad -1 \quad -1]$ for the relation between the five nodes, although they are rotated in our case.
For further exercising, we also want to compute the stiffness matrix for one of the boundary points; let's consider $P1$ for this case:

$$
\begin{aligned}
A_{11} &= \sum_k \int_{E_k} \nabla\varphi_3 \cdot \nabla\varphi_1 \\
&= \int_{E_1} \nabla\varphi_1 \cdot \nabla\varphi_1 + \int_{E_4} \nabla\varphi_1 \cdot \nabla\varphi_1 \\
&= \tfrac{1}{2} + \tfrac{1}{2} = 1 \\
A_{12} &= \sum_k \int_{E_k} \nabla\varphi_2 \cdot \nabla\varphi_1 \\
&= \int_{E_1} \nabla\varphi_2 \cdot \nabla\varphi_1 \\
&= 0 \\
A_{13} &= M_{31} = -1 \\
A_{14} &= \sum_k \int_{E_k} \nabla\varphi_4 \cdot \nabla\varphi_1 \\
&= 0 \\
A_{15} &= \sum_k \int_{E_k} \nabla\varphi_5 \cdot \nabla\varphi_1 \\
&= \int_{E_4} \nabla\varphi_5 \cdot \nabla\varphi_1 \\
&= 0
\end{aligned}
$$

(I) Exercise 3: Poisson Equation on a Triangular Grid

We solve the two-dimensional Poisson equation

$$
\begin{aligned}
u_{xx}(x,y) + u_{yy}(x,y) &= f(x,y) \in \Omega, && (20) \\
u(x,y) &= 0 \text{ at } \partial\Omega && (21)
\end{aligned}
$$

on a domain Ω.

(a) Give the weak form of (20).

(b) We solve (20) using finite elements on a triangular grid. Piecewise linear functions are used as test functions. Define these functions for the triangle displayed in the left Figure 4 and compute their gradients.

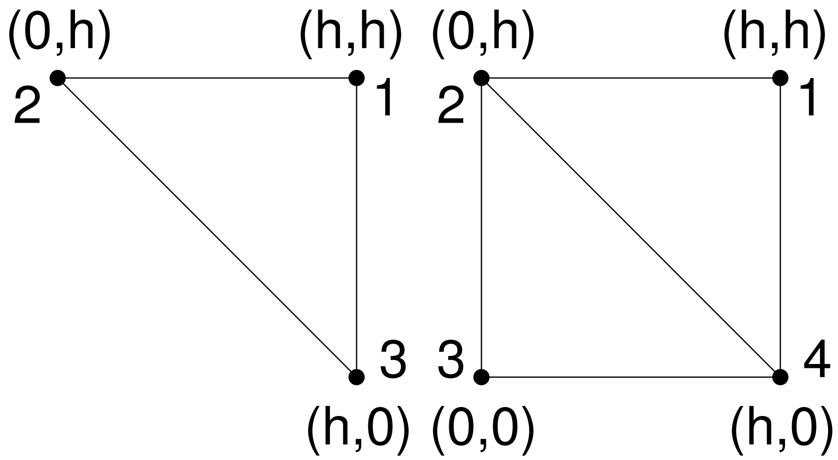

Figure 4: Left: Reference triangle. Right: Domain with two triangular elements.

(c) Compute the element stiffness matrix for the reference triangle. Test and ansatz space are the same in our example.

(d) Use the results from (c) to assemble the matrix for the grid consisting of two elements in the right Figure 4.

Hint: Consider only the given two elements without boundary conditions. Be careful with the numbering of nodes. The resulting two-element matrix should use the numbering given in the right Figure 4.

(e) Give a reason why you might want to assemble element stiffness matrices instead of a discretization stencil.

Solution:

(a) Multiply with test function, integrate, and apply Green's formula:

$$\begin{aligned}\int_\Omega v(x,y)\left(u_{xx}(x,y)+u_{yy}(x,y)\right)dxdy &= \int_\Omega v(x,y)f(x,y)dxdy, \text{ for all } v,\\ -\int_\Omega \left(v_x(x,y)u_x(x,y)+v_y(x,y)u_y(x,y)\right)dxdy &= \int_\Omega v(x,y)f(x,y)dxdy \text{ for all } v.\end{aligned}$$

(b) The nodal basis test functions are given by

$$\phi_1(x,y) = \frac{1}{h}x + \frac{1}{h}y - 1, \tag{22}$$

$$\phi_2(x,y) = -\frac{1}{h}x + 1, \tag{23}$$

$$\phi_3(x,y) = -\frac{1}{h}y + 1, \tag{24}$$

where ϕ_i is the basis function associated to node i of the triangle.

Derivatives of the basis functions:

$$\begin{aligned} \phi_{1,x}(x,y) &= \tfrac{1}{h}, & \phi_{1,y}(x,y) &= \frac{1}{h}, \\ \phi_{2,x}(x,y) &= -\tfrac{1}{h}, & \phi_{2,y}(x,y) &= 0, \\ \phi_{3,x}(x,y) &= 0, & \phi_{3,y}(x,y) &= -\frac{1}{h}. \end{aligned}$$

(c) Exploiting symmetries we compute matrix entries:

$$\begin{aligned} A_{1,1} &= -\int_0^h \int_y^h \left(\phi_{1,x}^2(x,y) + \phi_{1,y}^2(x,y)\right) dxdy = -\int_0^h \int_y^h \frac{2}{h^2} dxdy \\ &= -1, \end{aligned}$$

$$\begin{aligned} A_{1,2} &= -\int_0^h \int_y^h \left(\phi_{1,x}(x,y)\phi_{2,x}(x,y) + \phi_{1,y}(x,y)\phi_{2,y}(x,y)\right) dxdy \\ &= -\int_0^h \int_y^h -\frac{1}{h^2} dxdy \\ &= \frac{1}{2}, \\ A_{2,1} &= A_{1,2}, \end{aligned}$$

$$\begin{aligned} A_{1,3} &= -\int_0^h \int_y^h \left(\phi_{1,x}(x,y)\phi_{3,x}(x,y) + \phi_{1,y}(x,y)\phi_{3,y}(x,y)\right) dxdy \\ &= -\int_0^h \int_y^h -\frac{1}{h^2} dxdy \\ &= \frac{1}{2}, \\ A_{3,1} &= A_{1,3}, \end{aligned}$$

$$\begin{aligned} A_{2,2} &= -\int_0^h \int_y^h \left(\phi_{2,x}^2(x,y) + \phi_{2,y}^2(x,y)\right) dxdy = -\int_0^h \int_y^h \frac{1}{h^2} \\ &= -\frac{1}{2}, \end{aligned}$$

$$
\begin{aligned}
A_{2,3} &= -\int_0^h \int_y^h \left(\phi_{2,x}(x,y)\phi_{3,x}(x,y) + \phi_{2,y}(x,y)\phi_{3,y}(x,y)\right) dxdy \\
&= -\int_0^h \int_y^h 0 \\
&= 0, \\
A_{3,2} &= A_{2,3}, \\
A_{3,3} &= A_{2,2}.
\end{aligned}
$$

This results in the element matrix $\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$.

(d) 'Blow up' the element matrices and renumber to fit with the numbering of unknowns in the global two-element system:

Lower element:

Node number one in the reference element corresponds to number three in the global numbering, number two to number four, and number three to number two. Thus, add a zero line and column for node number one and reorder the remaining entries:

$$
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
$$

Upper element:

Node number one in the reference element corresponds to number one in the global numbering, number two to number two, and number three to number four. Thus, add a zero line and column for node number three and reorder the remaining entries:

$$
\begin{pmatrix} -1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}
$$

Add sum the two matrices:

$$
\begin{pmatrix} -1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & -1 \end{pmatrix}.
$$

(e) For an unstructured grid with irregular number of neighbors or an adaptive tree-structured grid with hanging nodes, the there is no stencil that can be applied to every node. In fact, one would have to store specialized stencils for every node. The element matrix, in contrast, stays the same up to transformations.