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Worksheet 13
Sample Solutions

Finite Element Methods

(H) Exercise 1: Convection-Diffusion Equations

Consider the convection-diffusion equation for temperature transport in a fluid which moves at constant velocity $v \in \mathbb{R}$:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $D \in \mathbb{R}^+$ denotes the diffusion constant of the fluid. The problem shall be solved on the unit interval with homogeneous Dirichlet conditions.

(a) Derive the weak formulation of the equation. Discretize in space by piecewise linear hat functions. Derive the semi-discrete set of equations and compute all coefficients. How can we categorize this set of equations?

(b) Perform mass lumping to facilitate the time discretization. Approximate the mass matrix $M_{ij} := \int \varphi_i \varphi_j dx$ by a diagonal matrix $\tilde{M}_{ij}$,

$$\tilde{M}_{ij} := \begin{cases} \sum\limits_j M_{ij} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Use the explicit Euler method to subsequently discretize the problem in time.

(c) Solve the problem with python for $D = 1.0$, $v = 1.0$, a mesh size $h = 1/10$ and a time step $\tau = 0.002, 0.02$. Use initial conditions $T = 1$ inside the domain (and homogeneous Dirichlet conditions at the boundaries).

Solution:

(a) Multiplying equation (1) by the test function $\varphi_i(x)$ and integrating in space yields:

$$\int_0^1 \frac{\partial T}{\partial t}\varphi_i(x)dx + v\int_0^1 \frac{\partial T}{\partial x}\varphi_i(x)dx = D\int_0^1 \frac{\partial^2 T}{\partial x^2}\varphi_i(x)dx \qquad \forall i \tag{3}$$

The right hand side can be transformed via integration by parts:

$$\int_0^1 \frac{\partial^2 T}{\partial x^2}\varphi_i(x)dx = \underbrace{\left[\frac{\partial T}{\partial x}\varphi_i(x)\right]_0^1}_{=0} - \int_0^1 \frac{\partial T}{\partial x}\frac{\partial \varphi_i}{\partial x}dx \tag{4}$$

The discrete solution of the temperature is written as linear combination of the basis functions, $T = \sum\limits_j a_j(t)\varphi_j(x)$. The basis functions are chosen to discretize space whereas the evolution over time is achieved via the coefficients $a_j(t)$. The arising finite element system of equations reads:

$$\begin{aligned} \sum_j \frac{\partial a_j(t)}{\partial t}\int_0^1 \varphi_i(x)\varphi_j(x)dx + v\cdot\sum_j a_j(t)\int_0^1 \frac{\partial \varphi_j(x)}{\partial x}\varphi_i(x)dx = \\ -D\sum_j a_j(t)\int_0^1 \frac{\partial \varphi_j(x)}{\partial x}\frac{\partial \varphi_i(x)}{\partial x}dx \quad \forall i \end{aligned} \tag{5}$$

To simplify the writing, we define the mass matrix M, the stiffness matrix A and the convective transport matrix C:

$$\begin{aligned} M_{ij} &:= \int_0^1 \varphi_i(x)\varphi_j(x)dx \\ A_{ij} &:= \int_0^1 \frac{\partial \varphi_j(x)}{\partial x}\frac{\partial \varphi_i(x)}{\partial x}dx \\ C_{ij} &:= \int_0^1 \frac{\partial \varphi_j(x)}{\partial x}\varphi_i(x)dx \end{aligned} \tag{6}$$

Our problem thus reads:

$$\sum_j M_{ij}\frac{\partial a_j(t)}{\partial t} + v\cdot\sum_j C_{ij}a_j(t) = -D\sum_j A_{ij}a_j(t) \quad \forall i \tag{7}$$

Now, we can insert the piecewise linear hat functions $\varphi_i(x)$ and evaluate the coefficients M_{ij}, A_{ij} and C_{ij}. Let h denote the mesh size of our one-dimensional grid. Integration of the local hat functions yields:

$$\begin{aligned} M_{ij} &:= \begin{cases} \frac{1}{6}h & |i-j| = 1 \\ \frac{2}{3}h \quad \text{if} & i = j \\ 0 & \text{otherwise} \end{cases} \\ A_{ij} &:= \begin{cases} -\frac{1}{h} & |i-j| = 1 \\ \frac{2}{h} \quad \text{if} & i = j \\ 0 & \text{otherwise} \end{cases} \\ C_{ij} &:= \begin{cases} \frac{1}{2} & j = i+1 \\ -\frac{1}{2} \quad \text{if} & j = i-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{8}$$

The set of equations from equation (7) is a *coupled system of ODEs*. As we can see from the form of the mass matrix M, a coupling of the time derivatives for the coefficients $a_{i-1}(t)$, $a_i(t)$ and $a_{i+1}(t)$ is currently enforced. This automatically does not allow for explicit time-stepping.

(b) We apply the mass lumping and so reduce the tridiagonal structure of the mass matrix to a diagonal matrix. We collect all contributions of a single row of the mass matrix M and store their sum in a diagonal matrix $\tilde{M}$:

$$\tilde{M}_{ii} = \sum_j M_{ij}, \quad \tilde{M}_{ij} = 0 \quad \text{for } i \neq j. \tag{9}$$

This procedure conserves the overall weight of a single matrix row and is therefore known as *mass lumping*. However, there is no coupling of basis functions included anymore. With respect to momentum or energy conservation, this implies that linear momentum or kinetic energy is conserved whereas angular momentum may not be preserved[1]. Considering the matrix M from the part (a), we obtain $\tilde{M}_{ii} = \sum_j M_{ij} = \frac{1}{6}h + \frac{2}{3}h + \frac{1}{6}h = h$. Inserting all matrix expressions including the mass lumping approach into equation (7) yields:

$$\begin{aligned}
& \tilde{M}_{ii}\tfrac{\partial a_i(t)}{\partial t} &=& -v \cdot \sum_j C_{ij}a_j(t) - D\sum_j A_{ij}a_j(t) & \forall i \\
\Leftrightarrow\ & h\tfrac{\partial a_i(t)}{\partial t} &=& -\tfrac{v}{2}(a_{i+1}(t) - a_{i-1}(t)) - \tfrac{D}{h}(-a_{i-1}(t) + 2a_i(t) - a_{i+1}(t)) & \forall i \\
\Leftrightarrow\ & \tfrac{\partial a_i(t)}{\partial t} &=& -\tfrac{v}{2h}(a_{i+1}(t) - a_{i-1}(t)) - \tfrac{D}{h^2}(-a_{i-1}(t) + 2a_i(t) - a_{i+1}(t)) & \forall i
\end{aligned} \tag{10}$$

Remark: we already dealt with convection-diffusion problems before, cf. Worksheet 8, Exercise 1. If we compare the right hand side of the equation above with the discrete version of the steady-state problem from Worksheet 8, Exercise 1(a), we can observe great similarity between both formulations in this special case (for $D = 1$).

Similar to previous discrete formulations for time-space partial differential equations, we obtained a (simplified) system of ordinary equations that can now be solved by any time-stepping method. We use the explicit Euler method and a time step τ:

$$\frac{\partial a_i(t)}{\partial t} = f(t) \rightarrow \frac{a_i(t+\tau) - a_i(t)}{\tau} = f(t) \leftrightarrow a_i(t+\tau) = a_i(t) + \tau f(t) \tag{11}$$

Inserting the explicit Euler method into equation (10) results in:

$$\begin{aligned}
a_i(t+\tau) &= a_i(t) - \tfrac{v\tau}{2h}(a_{i+1}(t) - a_{i-1}(t)) - \tfrac{D\tau}{h^2}(-a_{i-1}(t) + 2a_i(t) - a_{i+1}(t)) \\
a_i(t+\tau) &= \left(\tfrac{v\tau}{2h} + \tfrac{D\tau}{h^2}\right)a_{i-1}(t) + \left(1 - \tfrac{2D\tau}{h^2}\right)a_i(t) + \left(-\tfrac{v\tau}{2h} + \tfrac{D\tau}{h^2}\right)a_{i+1}(t)
\end{aligned} \tag{12}$$

(c) See ws12_ex1.py. For the large time step, instabilities occur. Several sources for these instabilities can be stated: first, the explicit Euler method yields restrictions onto the time step. This can be resolved by implicit time stepping schemes. Second, the structure

[1] For further information on mass lumping and properties of the arising matrices, you may check out Chapter 32 of http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/.

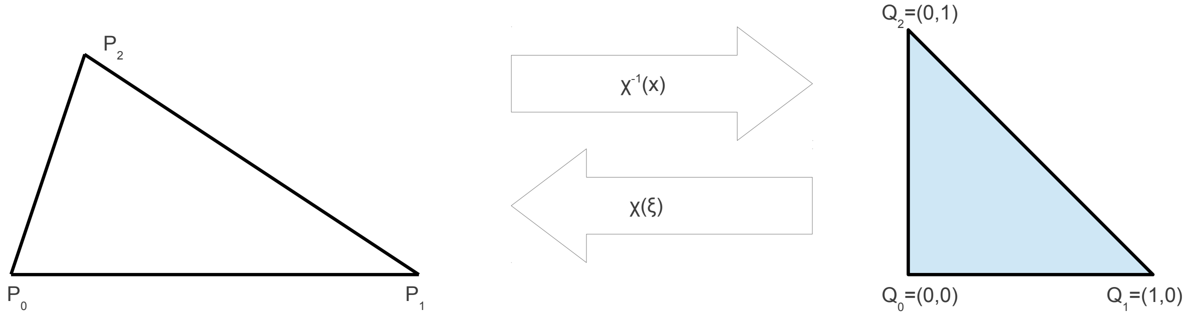

Figure 1: Coordinate mapping between a triangle of arbitrary shape (left) and a reference triangle (right).

of the matrix obtained from our finite element discretization resembles the case, when we discretized the convective term with central differences. From Neumann analysis of the finite difference version we know that it causes instability. To improve the stability we would need to change our basis functions such that the resulting matrix is more similar to the one we get when we use up-wind discretization for finite differences.

(H) Exercise 2: Reference Elements

In the following, we want to compute the mapping from an arbitrary triangle E onto a reference triangle E_{ref}, cf. Figure 1. This can be useful in several contexts, for example to simplify the integration procedures in the FE method or to prove error estimates for the respective finite elements and their basis functions.

(a) Define a transformation $\chi(\xi)$ which maps the coordinates ξ within the reference triangle E_{ref} (triangle on the right in Figure 1) onto the triangle E (triangle on the left in Figure 1). Use arbitrary coordinates P_0, P_1, $P_2 \in \mathbb{R}^2$ to define the transformation.

(b) Denote the corners of the reference triangle by $Q_0 = (0,0)^\top$, $Q_1 = (1,0)^\top$, $Q_2 = (0,1)^\top$. Define linear functions $\Phi_i(\xi)$, $i = 0,1,2$, on the reference triangle such that $\Phi_i(Q_j) = \delta_{ij}$, that is

$$\Phi_i(Q_j) := \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{13}$$

Compute the mass matrix $M_{ij}^{\text{ref}} := \int_{E_{\text{ref}}} \Phi_i(\xi)\Phi_j(\xi)d\xi$ of the reference element; you may use python for this purpose.

(c) Use the variable substitution to derive a formula which evaluates the mass matrix $M_{ij} = \int_E \phi_i(x)\phi_j(x)dx$ for an arbitrary triangle and its respective basis functions $\phi_i(x)$. The formula may only make use of the transformation $\chi(\xi)$ and the mass matrix M^{ref} of the reference triangle.

(d) Validate your formula from task (c) by computing the mass matrix of the reference element from Worksheet 11, Exercise 2. You may use python for this purpose.

Solution:

(a) Let $\xi \in E_{\text{ref}}$ denote a point inside the reference triangle. Then, we can define the mapping $\chi(\xi)$ as

$$\chi(\xi) := P_0 + B \cdot \xi, \quad B := (P_1 - P_0, P_2 - P_0) \in \mathbb{R}^{2\times 2}. \tag{14}$$

For $Q_0 = (0,0)^\top$, $Q_1 = (1,0)^\top$, $Q_2 = (0,1)^\top$, we thus obtain $\chi(Q_i) = P_i$, $i = 0,1,2$.

(b) The linear functions are given by

$$\begin{aligned} \Phi_0(\xi) &:= 1 - \xi_1 - \xi_2 \\ \Phi_1(\xi) &:= \xi_1 \\ \Phi_2(\xi) &:= \xi_2. \end{aligned} \tag{15}$$

The mass matrix evolves at

$$M^{\text{ref}} := \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix}, \tag{16}$$

see also ws12_ex2.py.

(c) Let $\phi_i(x)$ denote the linear functions on the general triangle E. By construction, these functions satisfy $\phi_i(P_j) = \delta_{ij}$. Now, we want to compute $M_{ij} = \int_E \phi_i(x)\phi_j(x)dx$. Using u-substitution, we can write the integral as

$$M_{ij} = \int_E \phi_i(x)\phi_j(x)dx = \int_{E_{\text{ref}}} \phi_i(\chi(\xi))\phi_j(\chi(\xi))|\det(\mathcal{J}(\xi))|d\xi \tag{17}$$

where $\mathcal{J}(\xi)$ denotes the Jacobi matrix of χ. The Jacobian $\mathcal{J}$ of a function $f(x) : \mathbb{R}^N \to \mathbb{R}^M$ is defined as

$$\mathcal{J} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix}. \tag{18}$$

Considering the linear mapping $\chi(\xi) := P_0 + B \cdot \xi$, we see that the Jacobian is given by $\mathcal{J}(\xi) = B$. On the current element E, the determinant $\det(B)$ is a constant, arising from

$$\det(B) = \begin{vmatrix} P_{1_1} - P_{0_1} & P_{2_1} - P_{0_1} \\ P_{1_2} - P_{0_2} & P_{2_2} - P_{0_2} \end{vmatrix} = (P_{1_1} - P_{0_1})(P_{2_2} - P_{0_2}) - (P_{2_1} - P_{0_1})(P_{1_2} - P_{0_2}). \tag{19}$$

We can hence move the determinant to the front of the integral expression:

$$M_{ij} = \int_{E_{\text{ref}}} \phi_i(\chi(\xi))\phi_j(\chi(\xi))|\det(B)|d\xi = |\det(B)| \int_{E_{\text{ref}}} \phi_i(\chi(\xi))\phi_j(\chi(\xi))d\xi \tag{20}$$

Consider the function $\phi_i(\chi(\xi))$ in more detail. If we insert Q_j, we see that

$$\phi_i(\chi(Q_j)) = \phi_i(P_0 + B \cdot Q_j) = \phi_i(P_j) = \delta_{ij}. \tag{21}$$

From this, we observe that the functions $\phi_i(\chi(\xi))$ and $\Phi_i(\xi)$ deliver the same results for all Q_j, $j = 0, 1, 2$. Since both functions are linear, they are further uniquely defined via these three points. As a result, both expressions must be identical,

$$\phi_i(\chi(\xi)) = \Phi_i(\xi). \tag{22}$$

The mass matrix for the general element E can hence be written as follows:

$$M_{ij} = |\det(B)| \int_{E_{\text{ref}}} \phi_i(\chi(\xi))\phi_j(\chi(\xi))d\xi = |\det(B)| \int_{E_{\text{ref}}} \Phi_i(\xi)\Phi_j(\xi)d\xi = |\det(B)| M_{ij}^{\text{ref}}. \tag{23}$$

(d) Computing the mass matrix for the linear basis functions on the reference element of Worksheet 11, Exercise 2 yields the matrix

$$M := \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} \tag{24}$$

We can further evaluate the determinant of the matrix B of this triangle:

$$\det(B) := |(P_2 - P_1 \quad P_3 - P_1)| = \left| \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right| = 2 \tag{25}$$

where P_i, $i = 1, 2, 3$ are the points illustrated in the respective figure of Exercise 26. Evaluating the formula from equation (23) shows the validity of our derivations:

$$M_{ij} = \begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix} = 2 \cdot \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{pmatrix} = |\det(B)| \cdot M_{ij}^{\text{ref}} \tag{26}$$

(H*) Exercise 3: Circus Tent

In order to analyze the stability of a circus tent, a two-dimensional simulation of the forces that act onto the tent construction is required. The tent construction is shown in Figure 2 on the left: from the middle of the tent, *spanning rods* branch out to the outer end of the circular tent. Each spanning rod has a length R. The spanning rods are homogeneously distributed; every pair of neighboring spanning rods is separated by an angle α. The ends of two spanning rods are held together by a *connecting rod*. Two spanning rods and one connecting rod form a triangular finite element. The element E_0 is chosen as reference element $E := E_0$; it is depicted in Figure 2 on the right.

(a) Determine linear basis functions $\Phi_i(x, y)$, $i = 0, 1, 2$, on the reference element such that $\Phi_i(Q_j) = \delta_{ij}$ for the points $Q_0 = (0, 0)$, $Q_1 = (R, 0)$, $Q_2 = (L, H)$. The basis functions may depend on L, R and H, respectively.

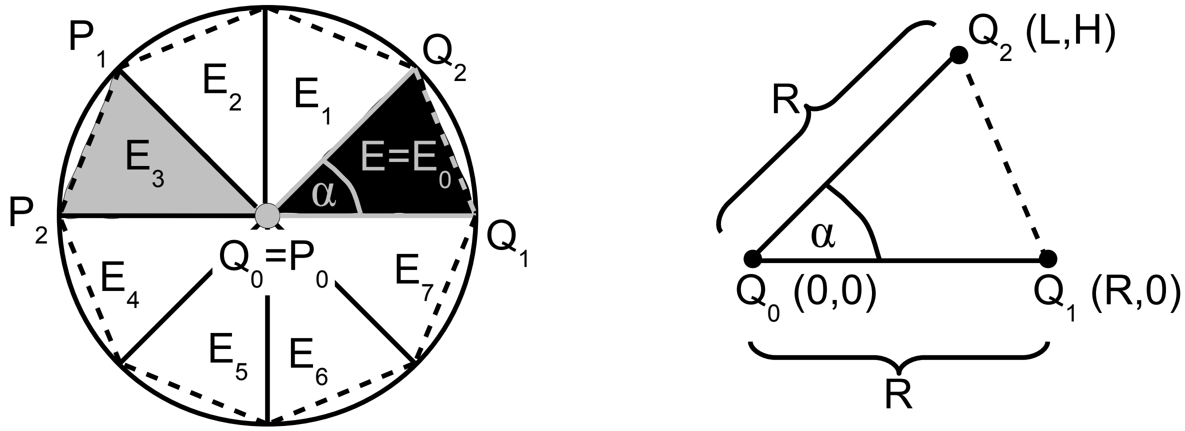

Figure 2: Left: circus tent. The tent is spanned by several rods branching out from the very middle. The rods are homogeneously distributed and are separated by an angle α. Additional rods (denoted by dashed lines) connect the outer ends of the spanning rods. Right: zoom into one of the elements that is spanned by two spanning rods and one connecting rod.

(b) Compute the entry

$$A^{\text{ref}}_{00} := \int_E \nabla\Phi_0 \cdot \nabla\Phi_0. \tag{27}$$

The final expression for A^{ref}_{00} should only depend on the angle α, i.e. $A^{\text{ref}}_{00} = A^{\text{ref}}_{00}(\alpha)$. It should not depend on L, R or H anymore.

(c) Determine a linear transformation rule $\chi^{(k)} : E \to E_k$ which maps the reference element $E = E_0$ onto any of the other elements E_k. The transformation should thus satisfy $\chi^{(k)}(Q_i) = P_i$, $i = 0, 1, 2$, where P_i corresponds to the respective vertex coordinates Q_i in the same consistent elementwise numbering scheme for element E_k (cf. Figure 2 on the left: the local vertex numbering is shown for elements E_0 and E_3.). The enumeration of the elements E_k is accomplished counter-clockwise starting from the reference element $E = E_0$ as illustrated in Figure 2 on the left.

Give an analytical expression for $\chi^{(k)}$ and show that the mass matrix entries (arising from Poisson-like PDE problems) are equal for all elements E_k, that is

$$\int_{E=E_0} \phi_i^{(0)}\phi_j^{(0)} = \int_{E_1} \phi_i^{(1)}\phi_j^{(1)} = \ldots = \int_{E_k} \phi_i^{(k)}\phi_j^{(k)} = \ldots \tag{28}$$

where $\phi_i^{(k)}$ denotes the i-th linear basis function ($i = 0, 1, 2$) which is locally defined on each element (k) as $\phi_i^{(k)}(P_j) = \delta_{ij}$.

Solution:

(a) The linear basis functions need to be of the form $\Phi_i(x, y) = c_0 + c_1 x + c_2 y$. The requirement $\Phi_i(Q_j) = \delta_{ij}$ yields three equations for the three coefficients c_0, c_1, c_2. Solving the equations for each $i = 0, 1, 2$ yields:

$$\begin{aligned}
\Phi_0(x,y) &= 1 - \frac{1}{R}x + \left(\frac{L}{RH} - \frac{1}{H}\right)y \\
\Phi_1(x,y) &= \frac{1}{R}x - \frac{L}{RH}y \\
\Phi_2(x,y) &= \frac{1}{H}y
\end{aligned} \tag{29}$$

(b) The gradient of $\Phi_0(x,y)$ reads:

$$\frac{\partial \Phi_0(x,y)}{\partial x} = -\frac{1}{R}, \quad \frac{\partial \Phi_0(x,y)}{\partial y} = \frac{L}{RH} - \frac{1}{H} \tag{30}$$

We thus obtain for the stiffness matrix entry A_{00}^{ref}:

$$\begin{aligned}
A_{00}^{\text{ref}} &= \int_E \nabla\Phi_0 \cdot \nabla\Phi_0 = \int_E \left(-\frac{1}{R}\right)^2 + \left(\frac{L}{RH} - \frac{1}{H}\right)^2 \\
&= \left(\frac{1}{R^2} + \frac{L^2}{R^2H^2} - \frac{2L}{RH^2} + \frac{1}{H^2}\right)\int_E 1
\end{aligned} \tag{31}$$

The surface of the element E is given by $|E| = \frac{1}{2}RH$. We obtain:

$$\begin{aligned}
A_{00}^{\text{ref}} &= \left(\frac{1}{R^2} + \frac{L^2}{R^2H^2} - \frac{2L}{RH^2} + \frac{1}{H^2}\right) \cdot \frac{1}{2}RH \\
&= \frac{H}{2R} + \frac{L^2}{2RH} - \frac{L}{H} + \frac{R}{2H} \\
&= \frac{H^2 + L^2 - 2LR + R^2}{2RH} \\
&\overset{H^2+L^2=R^2}{=} \frac{R}{H} - \frac{L}{H}
\end{aligned} \tag{32}$$

The expression $H^2 + L^2 = R^2$ just arises from Pythagoras' theorem, considering the reference element in Figure 2 on the right. We can finally use the trigonometric functions on the reference element to introduce the angle α. Since it holds $\sin(\alpha) = \frac{H}{R}$ and $\tan(\alpha) = \frac{H}{L}$, we can re-write equation (32) as follows:

$$A_{00}^{\text{ref}} = \frac{R}{H} - \frac{L}{H} = \frac{1}{\sin(\alpha)} - \frac{1}{\tan(\alpha)} \tag{33}$$

(c) All elements E_k are identical, but are rotated by an angle $\beta = k \cdot \alpha$. We can thus define a linear mapping as follows:

$$\chi^{(k)}(x,y) := \underbrace{\begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}}_{:=B^{(k)} \in \mathbb{R}^{2\times 2}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{34}$$

The mapping $\chi^{(k)}$ just rotates the reference triangle E onto E_k and it follows $\chi^{(k)}(Q_i) = P_i$ for $i = 0, 1, 2$. In Exercise 2, we have seen that the mass matrix entries $M_{ij}^{(k)}$ for an arbitrary triangle are derived from the mass matrix entries M_{ij}^{ref} of the reference triangle via

$$M_{ij}^{(k)} = |\det(B^{(k)})| M_{ij}^{ref}. \tag{35}$$

The determinant of rotation matrices is always 1 (and can be computed for the rotation matrix $\det(B^{(k)}) = 1$ as well). This shows that the mass matrix entries are equal for all elements (with respect to the same vertex numberings etc.).

(H*) Exercise 4: Stationary Convection-Diffusion Equations

The following differential equation for an unknown function $u(x, y)$ is defined on a square, $\Omega := (0, a) \times (0, b)$, with homogeneous Dirichlet conditions on all boundaries of the square:

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + q(x, y) \tag{36}$$

where $q(x, y)$ denotes a (known) source term.

We want to solve the problem on a Cartesian grid using the standard Galerkin procedure. For this purpose, we introduce locally *bilinear* basis functions which are defined on the reference element E given in Figure 3. The basis functions on the local reference element are given by

$$\begin{aligned} \varphi_0(x,y) &= (1-x)(1-y), \\ \varphi_1(x,y) &= x(1-y), \\ \varphi_2(x,y) &= (1-x)y, \\ \varphi_3(x,y) &= xy. \end{aligned} \tag{37}$$

For the corners P_0,..., P_3 of the reference element, cf. Figure 3, this yields $\varphi_i(P_j) = \delta_{ij}$ similar to the case of using piecewise linear basis functions on triangles.

(a) Derive the weak formulation of equation (36) for test functions $\psi(x, y)$ which belong to some function space V, $\psi(x, y) \in V$. No discretization of the function space V is required. Use integration by parts to transform the second-order derivative. Give a brief explanation why this transformation is helpful.

(b) Discretize your weak formulation using the basis functions $\varphi_i(x, y)$. Re-write the system in matrix-vector form $C \cdot \vec{u} = A \cdot \vec{u} + M \cdot \vec{q}$ with matrices

- C describing convection
- A describing diffusion
- M invoking sources terms.

Give a definition for each matrix entry C_{ij}, A_{ij} ,M_{ij}.

(c) Compute the contributions of the matrices A_{33} and C_{33} on the reference element.

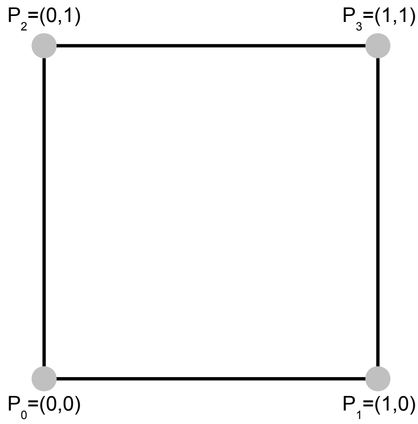

Figure 3: Cartesian grid cell for the application of bilinear ansatz functions.

Solution:

(a) Multiplying equation (36) by a test function $\psi(x,y)$ and integrating over the square yields:

$$\int\limits_{\Omega} \frac{\partial u}{\partial x}\psi(x,y)dxdy = \int\limits_{\Omega} \frac{\partial^2 u}{\partial x^2}\psi(x,y)dxdy + \int\limits_{\Omega} \frac{\partial^2 u}{\partial y^2}\psi(x,y)dxdy + \int\limits_{\Omega} q\psi(x,y)dxdy \quad \forall\psi \tag{38}$$

Integration by parts for the second-order derivative results in:

$$\begin{aligned}\int\limits_{\Omega} \frac{\partial u}{\partial x}\psi(x,y)dxdy = \underbrace{\left[\frac{\partial u}{\partial x}\psi(x,y) + \frac{\partial u}{\partial y}\psi(x,y)\right]_{\partial\Omega}}_{=0} - \int\limits_{\Omega} \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}dxdy \\ - \int\limits_{\Omega} \frac{\partial u}{\partial y}\frac{\partial \psi}{\partial y}dxdy + \int\limits_{\Omega} q\psi(x,y)dxdy \quad \forall\psi\end{aligned} \tag{39}$$

where the first term on the right hand side vanishes due to the homogeneous Dirichlet conditions. The remaining equation reads:

$$\int\limits_{\Omega} \frac{\partial u}{\partial x}\psi(x,y)dxdy = -\int\limits_{\Omega} \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x}dxdy - \int\limits_{\Omega} \frac{\partial u}{\partial y}\frac{\partial \psi}{\partial y}dxdy + \int\limits_{\Omega} q\psi(x,y)dxdy \quad \forall\psi \tag{40}$$

The integration by parts helps us to reduce the assumptions that u needs to fulfill. From the last equation, we can see that u needs to be differentiable once and must be integrable in some sense. We could thus remove the requirement for u to have a second-order derivative.

(b) Discretizing the function space V by $\varphi_i(x,y)$ yields:

$$\int\limits_{\Omega} \frac{\partial u}{\partial x}\varphi_i(x,y)dxdy = -\int\limits_{\Omega} \frac{\partial u}{\partial x}\frac{\partial \varphi_i}{\partial x}dxdy - \int\limits_{\Omega} \frac{\partial u}{\partial y}\frac{\partial \varphi_i}{\partial y}dxdy + \int\limits_{\Omega} q\varphi_i(x,y)dxdy \quad \forall\varphi_i \tag{41}$$

We can now write u and q as linear combinations of the functions $\varphi_i(x,y)$:

$$\begin{aligned} u(x,y) &= \sum_j u_j\varphi_j(x,y) \\ q(x,y) &= \sum_j q_j\varphi_j(x,y) \end{aligned} \tag{42}$$

Inserting this form of u, q into the discrete version of the weak formulation from above yields:

$$\begin{aligned} \sum_j u_j \int_\Omega \frac{\partial\varphi_j(x,y)}{\partial x}\varphi_i(x,y)dxdy &= -\sum_j u_j \int_\Omega \frac{\partial\varphi_j(x,y)}{\partial x}\frac{\partial\varphi_i(x,y)}{\partial x}dxdy \\ &- \sum_j u_j \int_\Omega \frac{\partial\varphi_j(x,y)}{\partial y}\frac{\partial\varphi_i(x,y)}{\partial y}dxdy \\ &+ \sum_j q_j \int_\Omega \varphi_j(x,y)\varphi_i(x,y)dxdy \quad \forall\varphi_i \end{aligned} \tag{43}$$

The matrix entries thus read:

$$\begin{aligned} C_{ij} &= \int_\Omega \frac{\partial\varphi_j(x,y)}{\partial x}\varphi_i(x,y)dxdy \\ A_{ij} &= -\int_\Omega \frac{\partial\varphi_j(x,y)}{\partial x}\frac{\partial\varphi_i(x,y)}{\partial x}dxdy - \int_\Omega \frac{\partial\varphi_j(x,y)}{\partial y}\frac{\partial\varphi_i(x,y)}{\partial y}dxdy \\ M_{ij} &= \int_\Omega \varphi_j(x,y)\varphi_i(x,y)dxdy \end{aligned} \tag{44}$$

(c) For the stiffness matrix, we need to first compute the partial derivatives of $\varphi_3(x,y)$:

$$\frac{\partial\varphi_3(x,y)}{\partial x} = y, \quad \frac{\partial\varphi_3(x,y)}{\partial y} = x \tag{45}$$

Now, we can solve the respective integral expression:

$$A_{33} = \int_\Omega (\nabla\varphi_3)^2 dxdy = \int_0^1\int_0^1 y^2 + x^2 dxdy = 1\cdot\left[\frac{1}{3}y^3\right]_0^1 + 1\cdot\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3} \tag{46}$$

For the convective term we do the analogous computation:

$$C_{33} = \int_\Omega \frac{\partial\varphi_3}{\partial x}\varphi_3 dxdy = \int_0^1\int_0^1 y\cdot(xy)dxdy = \left[\frac{1}{3}y^3\right]_0^1\cdot\left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{6} \tag{47}$$