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Worksheet 4
Problems

Continuous Models: Ordinary Differential Equations

Ordinary differential equations or ODEs are often used to describe *continuous problems* such as time-dependent phenomena. They contain a function $u(t)$ of one independent variable t and its derivatives du/dt, $f(t, u, du/dt, d^2u/dt^2, ..., d^nu/dt^n) = 0$. For ODE problems of the form

$$\frac{du(t)}{dt} = f(t) \cdot g(u),$$

the solution of the underlying problem can be determined via separation of variables:

1. For $g(u) \neq 0$, divide by $g(u)$:
$$\frac{1}{g(u)}\frac{du}{dt} = f(t)$$

2. Solve the integral equation:
$$\int \frac{1}{g(u)} du = \int f(t) dt$$

3. Solve the arising equation for u, i.e. write the equation as $u(t) = ...$.

(H) Exercise 1: Radioactive Decay

In radioactive decay, radioactive material is turned into more stable chemical elements. Let $N(t = 0)$ denote the original number of atoms. Then, $\lambda N(t)$ atoms $(0 < \lambda < 1)$ are expected to mutate over an infinitesimal time interval. This yields the following ordinary differential equation:

$$\frac{dN(t)}{dt} = -\lambda N(t) \tag{1}$$

(a) Solve the ODE (1) via separation of variables. How many solutions do we obtain? Under which assumption do we obtain a unique solution?

(b) The *half-life* t_H is the time after which 50% of the original material has mutated. Compute the half-life depending on an arbitrary choice of λ. How does the initial amount of atoms influence the half-life?

(c) General chemical reactions for a substance $N(t)$ can be modeled analogously to the radioactive decay. However, the substance may not only vanish, but it can also be created, e.g. due to other chemical reactions which happen simultaneously and don't necessarily depend on $N(t)$. How does the respective model for the substance $N(t)$ look like? Can you still solve the problem via separation of variables?

(H) Exercise 2: Eigenvalues of Differential Operators

Given an operator L, a real number λ is called *eigenvalue* of L, if

$$L(u) = \lambda u \tag{2}$$

for a non-zero function u. The function u is called *eigenfunction*.
Consider the case $L(u) := -\frac{d^2u}{dx^2}$.

(a) Show that all values $\lambda_k := (k\pi)^2$, $k \in \mathbb{N}$, are eigenvalues of L with corresponding eigenfunctions $u_k(x) := \sin(k\pi x)$, i.e. show that they solve the eigenvalue problem

$$L(u_k) = \lambda_k u_k, \quad u_k(0) = u_k(1) = 0 \tag{3}$$

on the unit interval.

(b) How does the problem from (a) translate to higher-dimensional problems, i.e. which functions u and scalars λ can you identify from the one-dimensional case which fulfill

$$-\sum_{d=1}^{D} \frac{\partial^2 u}{\partial x_d^2} = \lambda u, \quad u\|_{\partial\Omega} = 0 \tag{4}$$

where $\Omega = [0;1]^D$ is the unit hypercube and $\partial\Omega$ its boundary?

(I) Exercise 3: Population Models with One Species

Direction Fields

Name the corresponding models and provide the dynamics equations (basic equation without values of coefficients) for the direction fields provided in Figure 1.

Population Dynamics

Consider the following one-species population model

$$\frac{dp}{dt} = f(p) = -r\left(1 - \frac{p}{k}\right)\left(1 - \frac{p}{l}\right), \tag{5}$$

with $p(0) = p_0 \geq 0$ and $r > 0, 0 < l < k$.

(a) Compute the critical points. Are they stable or not?

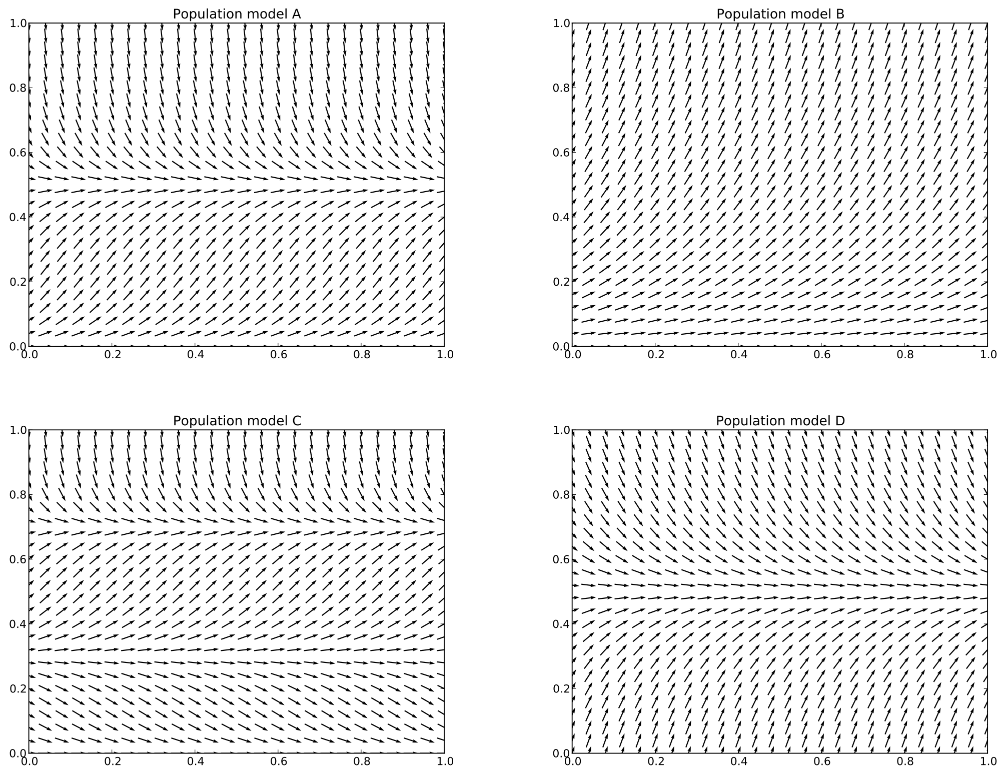

Figure 1: Direction fields for different population models with one species.

(b) Determine the limit $\lim_{t\to\infty} p(t)$ for

I) $0 \leq p_0 < l$,

II) $p_0 = l$,

III) $l < p_0 < k$,

IV) $p_0 = k$,

V) $p_0 > k$,

and sketch the trajectories of the solution $p(t)$ for each of the five cases.

(c) Name one reason why the model is not a realistic population model.