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Worksheet 6
Problems

Continuous Models: Ordinary Differential Equations

(H) Exercise 1: Critical Points and Direction Fields for Higher-Order ODEs

An ordinary differential equation of the form

$$f(t, y(t), dy/dt, d^2y/dt^2, \ldots, d^ny/dt^n) = 0 \tag{1}$$

is called ODE *of order* n, i.e. the index of the highest derivative corresponds to the order of the ODE. In order to facilitate the analysis of ODEs, every ODE of order $n > 1$ can be transformed into a system of ODEs of order $n = 1$ as explained below.

First, assume our ODE from equation (1) to have order $n > 1$. Then, we can introduce helper functions $y_0, \ldots, y_{n-1}$:

$$\begin{aligned} y_0(t) &:= y(t), \\ y_1(t) &:= \frac{dy(t)}{dt}, \\ &\vdots \\ y_{n-1} &:= \frac{d^{n-1}y(t)}{dt^{n-1}}. \end{aligned}$$

Second, using the helper functions, we can transform the ODE from equation (1) into the following system of ODEs:

$$\begin{aligned} &\frac{dy_0}{dt} = y_1, \\ &\frac{dy_1}{dt} = y_2, \\ &\quad\vdots \\ &f(t, y_0, \frac{dy_0}{dt}, \frac{dy_1}{dt}, \ldots, \frac{dy_{n-1}}{dt}) = 0. \end{aligned}$$

The latter system only consists of first-order derivatives with respect to the helper functions.

Complete the folowing tasks for the ODE

$$\frac{d^2y}{dt^2} = -y, \tag{2}$$

which is defined on an interval $t \in [0, \pi/2]$.

(a) Determine the order of this ODE.

(b) Transform the ODE into a system of first-order ODEs. Write the transformed system as

$$\begin{pmatrix} \frac{dy_0}{dt} \\ \vdots \\ \frac{dy_{n-1}}{dt} \end{pmatrix} = A \cdot \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3}$$

with a system matrix $A \in \mathbb{R}^{n \times n}$.

(c) Similar to the one-dimensional case, *homogeneous systems of ODEs*, that is ODEs which are of the form from equation (3) (without a constant vector b), can be solved analytically using the exponential function for matrices. In case of equation (3) and initial conditions $y_0(0) = c_0, y_1(0) = c_1, \ldots, y_{n-1}(0) = c_{n-1}$, the solution is given by:

$$\begin{pmatrix} y_0(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = \exp(A \cdot t) \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{4}$$

Use the results from the Worksheet 4 Exercise 4 to determine the general solution of equation (2). Which information is required to obtain a unique solution? Find at least two approaches to obtain a unique solution for the given ODE.

(d) Consider the ODE from equation (2) and its respective matrix-vector form (equation (3)). What can you say about its critical points and stability? Draw the direction field on $[-1;1] \times [-1;1]$.

(e) Write a python script which plots the direction field to validate your theoretical results from (d).

(f) Consider the modified ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t)$$

with $\mu \geq 0$. How does the parameter μ affect the critical point, stability and the direction field from (d)? Modify your python script for this purpose.

(g) Consider the ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t) + \frac{dy(t)}{dt}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. What happens to the critical point, stability and direction field in this case?

(I) Exercise 2: Nonlinear ODE – Logistic Function

The logistic function is given as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2. \tag{5}$$

(a) Calculate the critical points x^* and derive their stability for $b > 0$.

(b) Plot the value of the critical points as function of a/b for $a/b \in [-1, 1]$. Mark their stability.

(I) Exercise 3: Nonlinear Dynamic System – Linear Stability Analysis

The stability of a nonlinear system of ODEs can be determined by considering the linear behavior of this system in the neighborhood of the critical points. Without loss of generality, consider two first-order differential equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v(x,y) \end{aligned} \tag{6}$$

If x_0 and y_0 are critical points of the general system of ODEs (6), we can expand this system about (x_0, y_0)

$$\begin{aligned} \frac{\mathrm{d}(\delta x)}{\mathrm{d}t} &= \left.\frac{\partial u}{\partial x}\right|_{(x,y)=(x_0,y_0)} \delta x + \left.\frac{\partial u}{\partial y}\right|_{(x,y)=(x_0,y_0)} \delta y + \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(x,y)=(x_0,y_0)} \delta x \delta y + \dots \\ \frac{\mathrm{d}(\delta y)}{\mathrm{d}t} &= \left.\frac{\partial v}{\partial x}\right|_{(x,y)=(x_0,y_0)} \delta x + \left.\frac{\partial v}{\partial y}\right|_{(x,y)=(x_0,y_0)} \delta y + \left.\frac{\partial^2 v}{\partial x \partial y}\right|_{(x,y)=(x_0,y_0)} \delta x \delta y + \dots \end{aligned} \tag{7}$$

where we denoted $\delta x = x - x_0$ and $\delta y = y - y_0$.

The first order approximation then is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|_{(x,y)=(x_0,y_0)} \cdot \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = J \cdot \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{8}$$

The matrix J is called the Jacobian or stability matrix. By analysing the eigenvalues of this matrix in the same way as for linear systems we determine the stability and type of the critical points.

As an example consider a general competing species model

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 36x - 12x^2 - 2xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 30y - 6y^2 - 6xy \end{aligned} \tag{9}$$

where two species are denoted by $x \geq 0$ and $y \geq 0$.

To analyze the evolution of the species complete the following tasks:

(a) Find all the critical points of the dynamic system.

(b) Compute the Jacobian matrix of the dynamic system and analyze the stability and type of the critical points.

(c) Sketch a phase portrait or direction field of the given two species model, marking all the critical points on your sketch.

(H) Exercise 4: Nonlinear Dynamic System – Lyapunov Function Method

In some cases determining the linear behavior of a dynamic system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, where $x \in \mathbb{R}^n$, at a critical point x^* is not enough to characterize its stability. There are various methods to determine the stability of such a point. One of these requires finding a scalar Lyapunov function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ with the properties:

- $V(x) > 0, \quad \forall x \neq x^*$
- $V(x^*) = 0$

The existence of Lyapunov function for which $\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \nabla V(x) \cdot f(x) \leq 0, \quad \forall x \neq x^*$, guarantees the stability of the critical point x^*.

As an example consider the following two-dimensional dynamic system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x^2 - y\end{aligned} \tag{10}$$

(a) Plot the nullclines ($\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ or $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$) of the system and sketch the direction field.

(b) Compute the eigenvalues of the Jacobian matrix on the critical point of the system. Can you classify the critical point based on the eigenvalues?

(c) Use $V(x,y) = x^2 + y^2$ as a Lyapunov function to show that the critical point is stable.

(H*) Exercise 5: Discrete Dynamic System – Logistic Map

Let $x_{n+1} = f(x_n)$ be a map with

$$f(x) = ax(1-x), \quad a \geq 0 \tag{11}$$

(a) Determine the critical points of the map, i. e. the points x^* for which $f(x^*) = x^*$ holds. In addition, discuss their stability as a function of the parameter a.

(b) Determine the critical points of the map $f^{(2)}(x) := f(f(x))$.

Hint: $f(x)$ and $f^{(2)}(x)$ share two critical points. Make use of this and apply polynomial division.

(c) Sketch the stable critical points as a function of $0 \leq a \leq 4$.