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WS 2016/17
Scientific Computing
28.11.2016/30.11.2016

Worksheet 6
Sample Solutions

Continuous Models: Ordinary Differential Equations

(H) Exercise 1: Critical Points and Direction Fields for Higher-Order ODEs

An ordinary differential equation of the form

$$f(t, y(t), dy/dt, d^2y/dt^2, \dots, d^ny/dt^n) = 0 \tag{1}$$

is called ODE *of order n*, i.e. the index of the highest derivative corresponds to the order of the ODE. In order to facilitate the analysis of ODEs, every ODE of order $n > 1$ can be transformed into a system of ODEs of order $n = 1$ as explained below.

First, assume our ODE from equation (1) to have order $n > 1$. Then, we can introduce helper functions $y_0, \dots, y_{n-1}$:

$$\begin{aligned} y_0(t) &:= y(t), \\ y_1(t) &:= \frac{dy(t)}{dt}, \\ &\vdots \\ y_{n-1} &:= \frac{d^{n-1}y(t)}{dt^{n-1}}. \end{aligned}$$

Second, using the helper functions, we can transform the ODE from equation (1) into the following system of ODEs:

$$\begin{aligned} &\frac{dy_0}{dt} = y_1, \\ &\frac{dy_1}{dt} = y_2, \\ &\quad\vdots \\ &f(t, y_0, \frac{dy_0}{dt}, \frac{dy_1}{dt}, \dots, \frac{dy_{n-1}}{dt}) = 0. \end{aligned}$$

The latter system only consists of first-order derivatives with respect to the helper functions.

Complete the folowing tasks for the ODE

$$\frac{d^2y}{dt^2} = -y, \tag{2}$$

which is defined on an interval $t \in [0, \pi/2]$.

(a) Determine the order of this ODE.

(b) Transform the ODE into a system of first-order ODEs. Write the transformed system as

$$\begin{pmatrix} \frac{dy_0}{dt} \\ \vdots \\ \frac{dy_{n-1}}{dt} \end{pmatrix} = A \cdot \begin{pmatrix} y_0 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3}$$

with a system matrix $A \in \mathbb{R}^{n \times n}$.

(c) Similar to the one-dimensional case, *homogeneous systems of ODEs*, that is ODEs which are of the form from equation (3) (without a constant vector b), can be solved analytically using the exponential function for matrices. In case of equation (3) and initial conditions $y_0(0) = c_0, y_1(0) = c_1, \ldots, y_{n-1}(0) = c_{n-1}$, the solution is given by:

$$\begin{pmatrix} y_0(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = \exp(A \cdot t) \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{4}$$

Use the results from the Worksheet 4 Exercise 4 to determine the general solution of equation (2). Which information is required to obtain a unique solution? Find at least two approaches to obtain a unique solution for the given ODE.

(d) Consider the ODE from equation (2) and its respective matrix-vector form (equation (3)). What can you say about its critical points and stability? Draw the direction field on $[-1;1] \times [-1;1]$.

(e) Write a python script which plots the direction field to validate your theoretical results from (d).

(f) Consider the modified ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t)$$

with $\mu \geq 0$. How does the parameter μ affect the critical point, stability and the direction field from (d)? Modify your python script for this purpose.

(g) Consider the ODE

$$\frac{d^2y(t)}{dt^2} = -\mu \cdot y(t) + \frac{dy(t)}{dt}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. What happens to the critical point, stability and direction field in this case?

Solution:

(a) The index of the highest derivative in equation (2) is two. Hence, the order of the ODE is $n = 2$.

(b) We introduce two functions $y_0(t)$ and $y_1(t)$ as follows:

$$\begin{aligned} y_0(t) &= y(t) \\ y_1(t) &= \frac{dy(t)}{dt} \end{aligned}$$

From this, we obtain the following relation for the first derivatives of y_0, y_1:

$$\begin{aligned} \frac{dy_0(t)}{dt} &= \frac{dy(t)}{dt} &&= & y_1(t) &= 0 \cdot y_0(t) + 1 \cdot y_1(t) \\ \frac{dy_1(t)}{dt} &= \frac{d^2y(t)}{dt^2} &&= & -y(t) = -y_0(t) &= -1 \cdot y_0(t) + 0 \cdot y_1(t) \end{aligned}$$

Now, we can write our first-order system of ODEs in the matrix-vector form:

$$\begin{pmatrix} \frac{dy_0(t)}{dt} \\ \frac{dy_1(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix}$$

with matrix

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

(c) From Worksheet 4 Exercise 4 we know that the exponential $\exp(A \cdot t)$ for our matrix A from above has the form

$$\exp(A \cdot t) = I \cdot \cos(t) + A \cdot \sin(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

The general solution of our ODE hence looks as follows:

$$\begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

Different possibilities exist to determine a unique solution, for example:

- Fix the solution of $y(t) = y_0(t)$ *at both sides* of the defined interval, i.e. prescribe $y(0) := K_0$ and $y(\pi/2) := K_1$. Inserting these relations into the first equation of our system delivers $c_0 = K_0$, $c_1 = K_1$.
- Fix the solution of $y(t) = y_0(t)$ *at one side and also fix its derivative*. For this purpose, we can prescribe $y(0) := K_0$ and $dy(0)/dt = y_1(0) := K_1$. Inserting these relations into the solution of the ODE system from above yields $c_0 = K_0$ and $c_1 = K_1$.

(d) The eigenvalues of the underlying matrix

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

are given by $\lambda = \pm i$. The eigenvalue is complex and its real part is zero. Following the table on "Stability of Linear Systems" (cf. Slide 27 of the lecture population2.pdf), the system is therefore expected to be stable and to have a "centered" critical point. Since A has full rank and since our system is homogeneous, the critical point is given by $A^{-1}\vec{0} = \vec{0}$.

(e) See ws6_ex1.py.

(f) See ws6_ex1.py. The eigenvalues arise in this case to be $\lambda = \pm\sqrt{\mu}i$. The stability is – according to the overview from the lecture – still guaranteed. The matrix A has full rank for all $\mu \neq 0$; the critical point remains at $\vec{0}$ in this case. The direction field has an ellipsoidal shape.

For $\mu = 0$, the ODE reads $d^2y(t)/dt^2 = 0$ and has the solution $y(t) = c \cdot t + d$ with constants $c, d \in \mathbb{R}$. The only eigenvalue of A is $\lambda = 0$ in this case.

(g) See ws6_ex1.py. Re-writing the ODE as a system of first-order ODEs yields a homogeneous system $(y_0(t), y_1(t))^\top = A \cdot (y_0(t), y_1(t))^\top$ with matrix

$$A := \begin{pmatrix} 0 & 1 \\ -\mu & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_{0,1} = \dfrac{1 \pm \sqrt{1-4\mu}}{2}$. Since $\mu \neq 0$, the inverse of A exists and is given by

$$A^{-1} := \frac{1}{\mu} \begin{pmatrix} 1 & -1 \\ \mu & 0 \end{pmatrix}.$$

For

- $\mu = 0.25$, we have only one real eigenvalue $\lambda_0 = \lambda_1 = 0.5 > 0$. From the overview in the lecture, we expect the system to have a single node as critical point and unstable behaviour. Since the system is homogeneous, the critical point is again given by $\vec{0}$.
- $\mu < 0.25$, we have two real-valued eigenvalues. If
  - $0 < \mu < 0.25$, all eigenvalues are bigger than zero $\Rightarrow$ critical point is a node, system is unstable
  - $\mu < 0$, one eigenvalue is bigger and one eigenvalue is smaller than zero $\Rightarrow$ critical point is a saddle point, system is unstable
- $\mu > 0.25$, we have complex eigenvalues, $\lambda_{0,1} = \frac{1}{2} \pm \sqrt{4\mu - 1}i$. Since the real part of the eigenvalues is always bigger than zero, we expect a spiral point and unstable behavior.

(I) Exercise 2: Nonlinear ODE – Logistic Function

The logistic function is given as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2. \tag{5}$$

(a) Calculate the critical points x^* and derive their stability for $b > 0$.

(b) Plot the value of the critical points as function of a/b for $a/b \in [-1, 1]$. Mark their stability.

Solution:

(a) For the critical points x^* the time derivative vanishes.

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x^*} = 0$$

This holds true at the points $x_1^* = 0$ and $x_2^* = \frac{a}{b}$.

To determine the stability of these points one has to calculate the spatial derivative of $\frac{dx}{dt}$. If the real part of the derivative is less than zero at the critical points, it is stable, otherwise it is unstable.

$$\mathrm{Re}\left(\frac{\partial}{\partial x}\frac{dx}{dt}\right) \begin{cases} < 0: & \text{stable} \\ > 0: & \text{unstable} \\ = 0: & \text{both possible} \end{cases}$$

The spatial derivative of the logistic function is $\frac{\partial}{\partial x}\frac{dx}{dt} = a - 2bx$. Thus the critical point $x_1^* = 0$ is stable for $a < 0$ and unstable for $a > 0$. For $a = 0$ equation (5) can be rewritten as $\frac{dx}{dt} = -bx^2$. The point $x^* = 0$ then is a saddle point. For $x_2^* = \frac{a}{b}$ the derivative becomes $\frac{\partial}{\partial x}\frac{dx}{dt} = -a$. The point is stable when $a > 0$, unstable for $a < 0$ and is a saddle point for $a = 0$.

(b) Figure 1 shows the bifurcation diagram for the logistic function. Stable points are marked green, unstable red. Remark: in the bifurcation point $a = 0$ the two critical points collide and swap their stability. This behavior is called transcritical bifurcation.

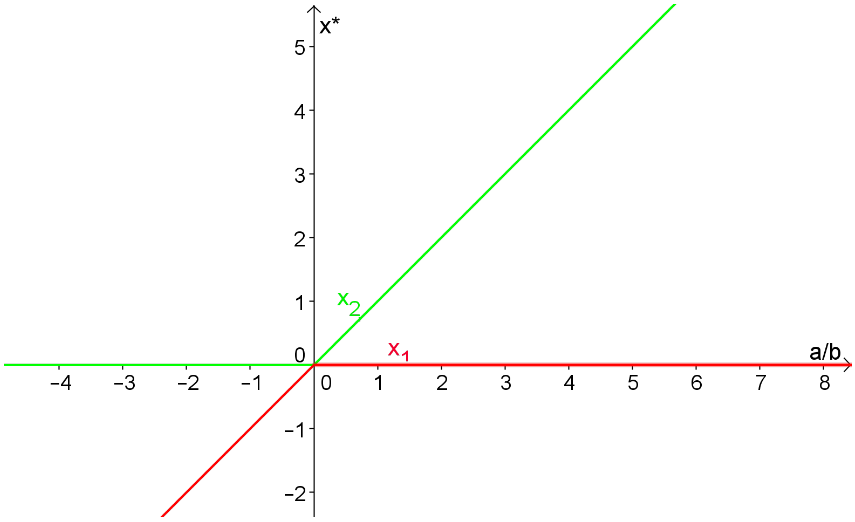

Figure 1: Bifurcation diagram of the continuous logistic equation. Stable points are marked green, unstable red.

(I) Exercise 3: Nonlinear Dynamic System – Linear Stability Analysis

The stability of a nonlinear system of ODEs can be determined by considering the linear behavior of this system in the neighborhood of the critical points. Without loss of generality, consider two first-order differential equations:

$$\begin{aligned} \frac{dx}{dt} &= u(x, y) \\ \frac{dy}{dt} &= v(x, y) \end{aligned} \tag{6}$$

If x_0 and y_0 are critical points of the general system of ODEs (6), we can expand this system about (x_0, y_0)

$$\begin{aligned}
\frac{\mathrm{d}(\delta x)}{\mathrm{d}t} &= \left.\frac{\partial u}{\partial x}\right|_{(x,y)=(x_0,y_0)} \delta x + \left.\frac{\partial u}{\partial y}\right|_{(x,y)=(x_0,y_0)} \delta y + \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(x,y)=(x_0,y_0)} \delta x \delta y + \ldots \\
\frac{\mathrm{d}(\delta y)}{\mathrm{d}t} &= \left.\frac{\partial v}{\partial x}\right|_{(x,y)=(x_0,y_0)} \delta x + \left.\frac{\partial v}{\partial y}\right|_{(x,y)=(x_0,y_0)} \delta y + \left.\frac{\partial^2 v}{\partial x \partial y}\right|_{(x,y)=(x_0,y_0)} \delta x \delta y + \ldots
\end{aligned} \tag{7}$$

where we denoted $\delta x = x - x_0$ and $\delta y = y - y_0$.

The first order approximation then is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|_{(x,y)=(x_0,y_0)} \cdot \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = J \cdot \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{8}$$

The matrix J is called the Jacobian or stability matrix. By analysing the eigenvalues of this matrix in the same way as for linear systems we determine the stability and type of the critical points.

As an example consider a general competing species model

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 36x - 12x^2 - 2xy \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 30y - 6y^2 - 6xy
\end{aligned} \tag{9}$$

where two species are denoted by $x \geq 0$ and $y \geq 0$.

To analyze the evolution of the species complete the following tasks:

(a) Find all the critical points of the dynamic system.

(b) Compute the Jacobian matrix of the dynamic system and analyze the stability and type of the critical points.

(c) Sketch a phase portrait or direction field of the given two species model, marking all the critical points on your sketch.

Solution:

(a) To find the critical points we solve the system of equations with all time derivatives set to zero

$$\begin{aligned}
x(36 - 12x - 2y) &= 0 \\
y(30 - 6y - 6x) &= 0
\end{aligned}$$

The elementary solution of this system of equations is $(x, y)_1 = (0, 0)$.

Next, we set $x = 0$ and solve $30 - 6y = 0$ to find y, what gives $(x, y)_2 = (0, 5)$.

Similarly, for $y = 0$ and $36 - 12x = 0$ we obtain $(x, y)_3 = (3, 0)$.

Finally, to find the last root of the system we solve the following linear system

$$\begin{aligned}
12x + 2y &= 36 \\
6y + 6x &= 30
\end{aligned}$$

which yields $(x, y)_4 = (2.6, 2.4)$.

(b) The Jacobian matrix takes the form

$$J = \begin{pmatrix} 36 - 24x - 2y & -2x \\ -6y & 30 - 12y - 6x \end{pmatrix}$$

Next, we analyze the stability and type of the critical points by computing eigenvalues of the Jacobian matrix at this points.

- Point $(x, y)_1 = (0, 0)$:

$$J(0,0) = \begin{pmatrix} 36 & 0 \\ 0 & 30 \end{pmatrix}$$

The eigenvalues of this Jacobian are both positive $\lambda_1 = 36$ and $\lambda_2 = 30$, so this is an unstable node.

- Point $(x, y)_2 = (0, 5)$:

$$J(0,5) = \begin{pmatrix} 26 & 0 \\ -30 & -30 \end{pmatrix}$$

The eigenvalues of the matrix are $\lambda_1 = 26 > 0$ and $\lambda_2 = -30 < 0$, so we have a saddle point.

- Point $(x, y)_3 = (3, 0)$:

$$J(3,0) = \begin{pmatrix} -36 & -6 \\ 0 & 12 \end{pmatrix}$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -36 < 0$, $\lambda_2 = 12 > 0$ and we obtain another saddle point.

- Point $(x, y)_4 = (2.6, 2.4)$:

$$J(2.6, 2.4) = \frac{2}{5} \begin{pmatrix} -78 & -13 \\ -36 & -36 \end{pmatrix}$$

The characteristic polynomial takes the form $\mu^2 + 6 \cdot 19\mu - 36 \cdot 65 = 0$, where $\mu = 2\lambda/5$. The roots of this polynomial are $\mu_{1,2} = -57 \left(1 \pm \sqrt{\frac{101}{361}}\right)$ and we have two negative eigenvalues $\lambda_{1,2} < 0$ and the critical point is a stable node.

(c) The direction field for the system of ODEs (9) is shown in Figure 2.

(H) Exercise 4: Nonlinear Dynamic System – Lyapunov Function Method

In some cases determining the linear behavior of a dynamic system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$, where $x \in \mathbb{R}^n$, at a critical point x^* is not enough to characterize its stability. There are various methods to determine the stability of such a point. One of these requires finding a scalar Lyapunov function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ with the properties:

- $V(x) > 0, \quad \forall x \neq x^*$
- $V(x^*) = 0$

The existence of Lyapunov function for which $\frac{\mathrm{d}V(x)}{\mathrm{d}t} = \nabla V(x) \cdot f(x) \leq 0, \quad \forall x \neq x^*$, guarantees the stability of the critical point x^*.

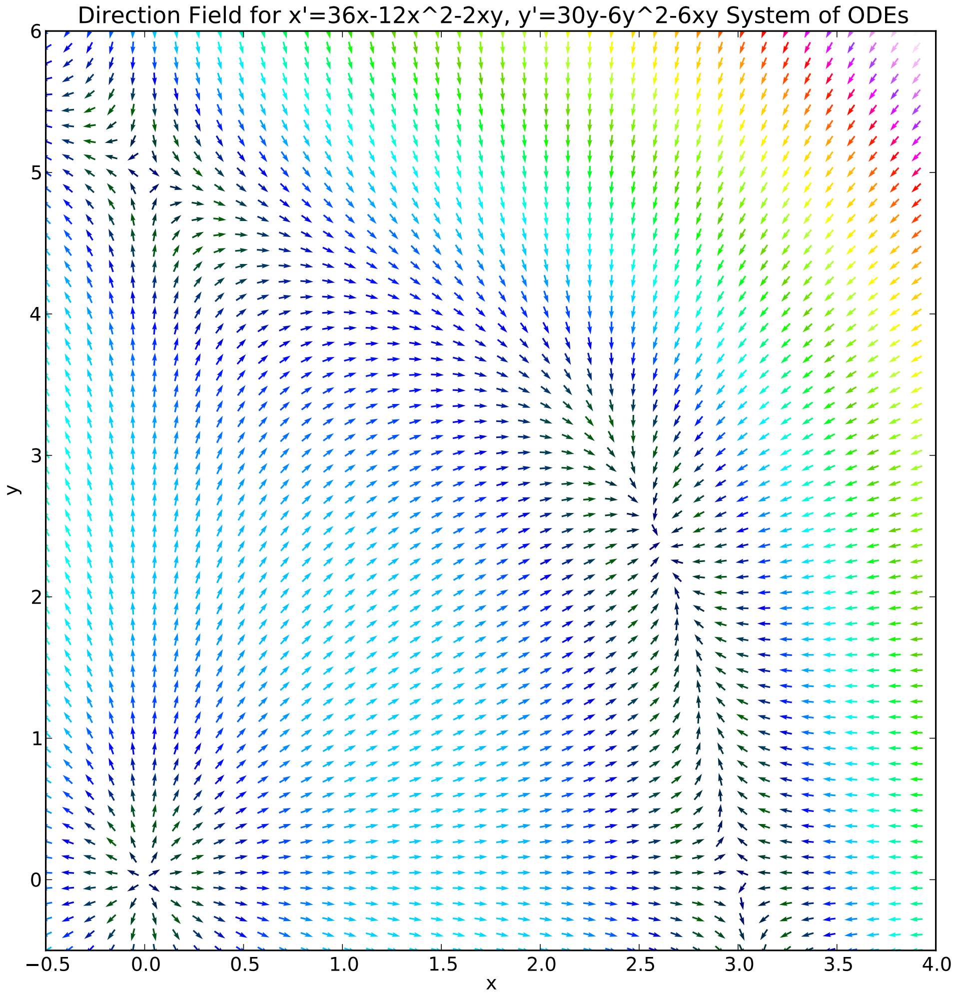

Figure 2: Direction field of the nonlinear system of ODEs (9)

As an example consider the following two-dimensional dynamic system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x^2 - y\end{aligned} \tag{10}$$

(a) Plot the nullclines ($\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ or $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$) of the system and sketch the direction field.

(b) Compute the eigenvalues of the Jacobian matrix on the critical point of the system. Can you classify the critical point based on the eigenvalues?

(c) Use $V(x, y) = x^2 + y^2$ as a Lyapunov function to show that the critical point is stable.

Solution:

(a) For the nullclines at least one of the time derivatives has to vanish.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Leftrightarrow x = 0 \vee y = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Leftrightarrow y = x^2$$

Figure 3 shows the nullclines and the vectorfield of the system. Note the critical point at $(x, y) = (0, 0)$.

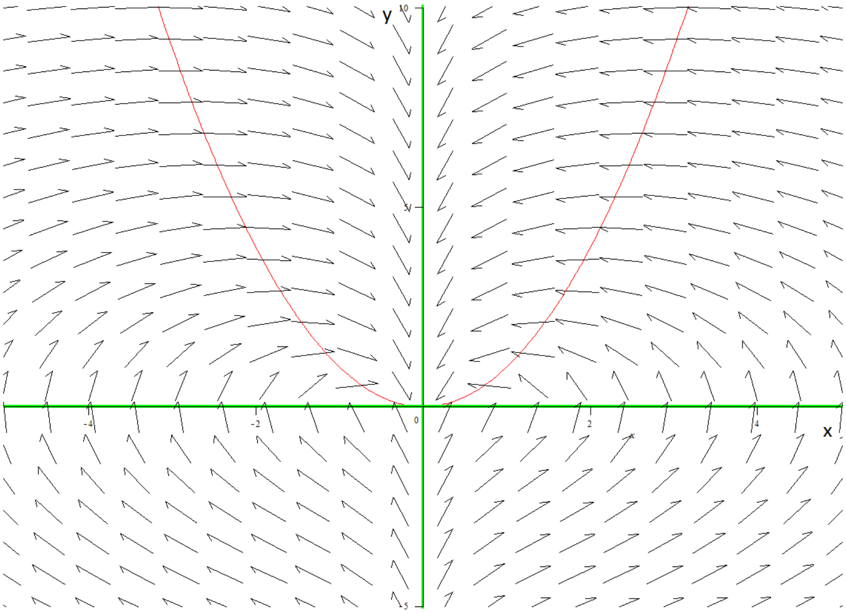

Figure 3: Vectorfield and nullclines ($\frac{\mathrm{d}y}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$) of the problem

(b) The Jacobian of the system is given by

$$J = \begin{pmatrix} -y & -x \\ 2x & -1 \end{pmatrix}\Bigg|_{x^*=(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$.

It is not possible to determine the stability just from the eigenvalues, since the eigenvalues need to be less than zero to determine the stability with certainty.

(c) The time derivative of the Lyapunov function V is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(x, y) &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= 2x(-xy) + 2y(x^2 - y) \\ &= -2y^2 \leq 0 \quad \forall x \neq x^*\end{aligned}$$

This is sufficient to prove the stability of the critical point. Note that even though $\frac{\mathrm{d}V(x,0)}{\mathrm{d}t} = 0$, since $\frac{\mathrm{d}y}{\mathrm{d}t} \neq 0 \,\forall y = 0, x \neq 0$, y will not remain zero and the function V gets smaller over time. The value of the function V thus steadily decreases and reaches the minimum at $x = x^*$.

(H*) Exercise 5: Discrete Dynamic System – Logistic Map

Let $x_{n+1} = f(x_n)$ be a map with

$$f(x) = ax(1-x), \quad a \geq 0 \tag{11}$$

(a) Determine the critical points of the map, i. e. the points x^* for which $f(x^*) = x^*$ holds. In addition, discuss their stability as a function of the parameter a.

(b) Determine the critical points of the map $f^{(2)}(x) := f(f(x))$.

Hint: $f(x)$ and $f^{(2)}(x)$ share two critical points. Make use of this and apply polynomial division.

(c) Sketch the stable critical points as a function of $0 \leq a \leq 4$.

Solution:

(a) For critical points the relation $f(x^*) = x^*$ holds. In the trivial case $a = 0$ the map reads $x_{n+1} = 0$, i. e. there is one stable critical point $x^* = 0$. Otherwise we have

$$f(x^*) = ax^*(1-x^*) \stackrel{!}{=} x^* \Leftrightarrow x^*[-ax^* + (a-1)] = 0$$

and therefore $x_1^* = 0$ and $x_2^* = \frac{a-1}{a}$.

To determine the stability of the critical points we look at $f'(x^*)$. If $|f'(x^*)| < 1$, the considered critical point is stable and for $|f'(x^*)| > 1$ the critical point is unstable. For the logistic map $f'(x) = a[1-2x]$. We already considered the trivial case $a = 0$. Hence we assume $a \neq 0$.

For $x_1^* = 0$, we have $f'(0) = a$, which means we have a stable critical point at $x_1^* = 0$ for $0 < a < 1$ and an unstable critical point for $a > 1$.

On the other hand, we have $f'(x_2^*) = 2 - a$ for $x_2^* = \frac{a-1}{a}$. If $1 < a < 3$ we obtain $|f'(x_2^*)| < 1$ and therefore a stable critical point. The choices $0 < a < 1$ or $a > 3$ result in $|f'(x_2^*)| > 1$ and an unstable critical point.

In total there are three different cases.

- $0 < a < 1$: $x_1^* = 0$ stable, $x_2^* = \frac{a-1}{a}$ unstable
- $1 < a < 3$: $x_1^* = 0$ unstable, $x_2^* = \frac{a-1}{a}$ stable
- $a > 3$: $x_1^* = 0$ unstable, $x_2^* = \frac{a-1}{a}$ unstable

For the parameter values $a = 1$ and $a = 3$ the stability properties of the critical points and therefore the behaviour of the system changes. This is called bifurcation.

(b) The critical points of $x_{n+1} = f(f(x_n))$ are defined by $x^* = f(f(x^*))$.

$$\begin{aligned} x^* &= f(f(x^*)) \\ &= af(x^*)(1 - f(x^*)) \\ &= a(ax^*(1-x^*))(1 - ax^*(1-x^*)) \end{aligned}$$

There are four solutions (critical points) of this equation because the corresponding polynomial is of degree four. For the critical points x_1^* and x_2^* of $x_{n+1} = f(x_n)$ it holds

$$f(f(x_{1,2}^*)) = f(x_{1,2}^*) = x_{1,2}^*.$$

Hence, critical points of $x_{n+1} = f(x_n)$ are also critical points of $x_{n+1} = f(f(x_n))$. We can use this result to transform the fourth order polynomial equation with the help of polynomial division into a second order polynomial equation, this results in the new critical points

$$x_{3,4}^* = \frac{-(a^3+a^2) \pm \sqrt{(a^3+a^2)^2 - 4(-a^3)(-a^2-a)}}{-2a^3}.$$

Those are real for $a > 3$ and fulfil the following relations

$$f(x_3^*) = x_4^* \quad \text{and} \quad f(x_4^*) = x_3^*.$$

(c) For $0 < a < 1$ only the critical point x_1^* is stable. At $a = 1$ the critical points exchange their stability properties (transcritical bifurcation) and the critical point x_2^* is stable and plotted in Figure 4. At $a = 3$ the critical points of $x_{n+1} = f(f(x_n))$ (see (b)) become real-valued. The simple logistic map shows chaotic behaviour for higher a.

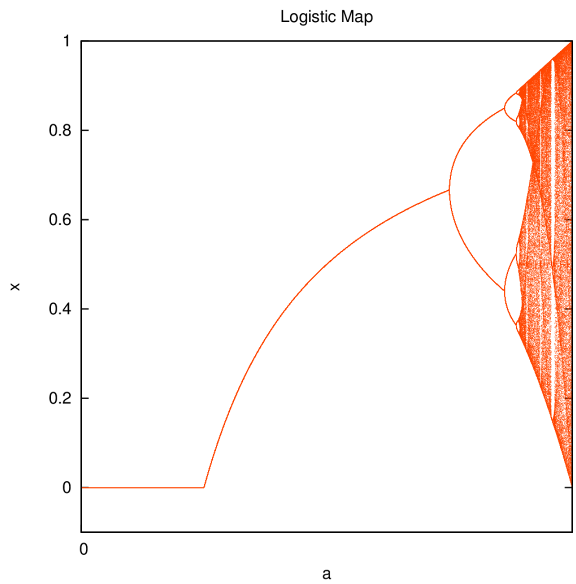

Figure 4: Stable critical points of the logistic map over the parameter a.