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Worksheet 9
Problems

Partial Differential Equations

Jacobi Method

An iterative method to solve linear systems of equations $Ax = b$ with $A \in \mathbb{R}^{N\times N}$, $b \in \mathbb{R}^N$ is given by the Jacobi method. Starting from an initial vector $x^{(0)}$, the iteration procedure reads:

$$x_i^{(n+1)} = \frac{1}{A_{ii}}\left(b_i - \sum_{j\neq i} A_{ij}x_j^{(n)}\right), \quad i = 1, ..., N$$

Convergence of the Jacobi Method: Diagonal dominance

A matrix $A \in \mathbb{R}^{N\times N}$ is *diagonally dominant* if the following inequality holds for all $i \in \{1, ..., N\}$:

$$|A_{ii}| \geq \sum_{j\neq i} |A_{ij}| \tag{1}$$

Diagonal dominance is an important property of matrices and helps to show that for example the Jacobi iteration scheme for $Ax = b$ converges:

- If A is strictly diagonally dominant, that is we have a "$>$" in equation (1) for all rows i, then the Jacobi iteration converges.
- Let A be diagonally dominant and have at least one row with "$>$" in equation (1). If A is irreducible, then the Jacobi method converges.

We do not want to dive deeper into mathematics here and, thus, will not completely define what irreducibility means (we may discuss the respective definition during the exercise class). However, it should be noted that a matrix is irreducible if the matrix is tridiagonal and only has non-vanishing main- and subdiagonal entries, that is

$$A_{ij} \neq 0 \quad \forall |i - j| \leq 1.$$

(H) Exercise 1: Convection-Diffusion Systems

Consider the one-dimensional differential equation

$$-\frac{d^2u(x)}{dx^2} + v \cdot \frac{du(x)}{dx} = f(x), \qquad x \in (0,1) \tag{2}$$

with velocity $v \in \mathbb{R}$ and Dirichlet boundary conditions $u(0) = c_0, u(1) = c_1$.
This equation models the transport of a quantity u in a fluid when the fluid is assumed to move at constant velocity v.

(a) Set up a *finite difference* scheme using

- the standard second-order discretization of the diffusive term (that is the second-order derivative)
- a symmetric second-order discretization for the convective term (that is the first-order derivative).

Write down the formulation for a single row of the arising linear system of equations

$$\sum_j A_{ij} u_j = b_i$$

with $u_j := u(j \cdot h)$, mesh size h and right-hand side b_i.

(b) Formulate the Jacobi relaxation to solve this system. Under which conditions does the Jacobi method converge?

(c) Replace the second-order discretization of the first-order derivative by a first-order one-sided discretization. Re-formulate the Jacobi relaxation for this case. Under which conditions can we expect convergence now?

(H) Exercise 2: Runge-Kutta for the Heat Equation

The following partial differential equation describes the distribution of the temperature T in a stick:

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \qquad x \in (0,1)$$

The constant D is the thermal diffusivity and describes how fast the temperature can diffuse within the stick. We further assume that the temperature of the stick at the outer ends is known, that is $T(t, x = 0) = T_0$, $T(t, x = 1) = T_N$.

(a) Review from the lecture: Apply the symmetric finite difference approximation for the second-order spatial derivative similar to exercise 1 (a). Which kind of differential equations remains?

(b) Discretize the new differential equations from (a) using the method of Heun.

(c) Write a python script that solves the discrete problem with $D = 1$, a time step $\tau = 0.001$, an initial temperature distribution $T(t = 0, x) = 0$ in the inner part of the stick and temperature values $T_0 = 0$, $T_1 = 1$. Use a mesh sizes $h = 1/20$ for the spatial discretization and plot the result after 1, 10, 100 and 1000 time steps.

(I) Exercise 3: Discretization of the Laplace Equation

Consider the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad (0;1)^2, \tag{3}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \vec{n}} = 0 \quad \text{at} \quad \{0\} \times [0;1] \cup \{1\} \times [0;1] \cup [0;1] \times \{0\} \cup [0;1] \times \{1\}$$

and the following finite difference discretization on a square grid, see Figure 1.

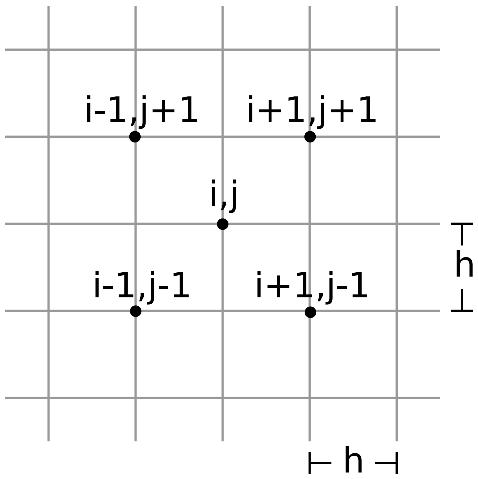

Figure 1: Finite difference discretization on a square grid.

$$\Delta u(x_{i,j}) \approx \frac{u_{i-1,j-1} + u_{i-1,j+1} - 4u_{i,j} + u_{i+1,j-1} + u_{i+1,j+1}}{2h^2} \tag{4}$$

at all inner grid points $x_{i,j}$. For the first order derivative on the boundary nodes use the central difference discretization:

$$\begin{aligned} u_{x_1}(x_{i,j}) &\approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} \\ u_{x_2}(x_{i,j}) &\approx \frac{u_{i,j+1} - u_{i,j-1}}{2h} \end{aligned} \tag{5}$$

(a) Is this discretization consistent?

(b) Does the discretization scheme lead to a unique physical solution? Hint: try to find an oscillating pattern in the sample grid in Figure 2, which satisfies the boundary condition and is annihilated by the discrete operator.

Remark: the solution of the continuous problem is unique only up to a constant function, but oscillations cannot occur in the exact solution.

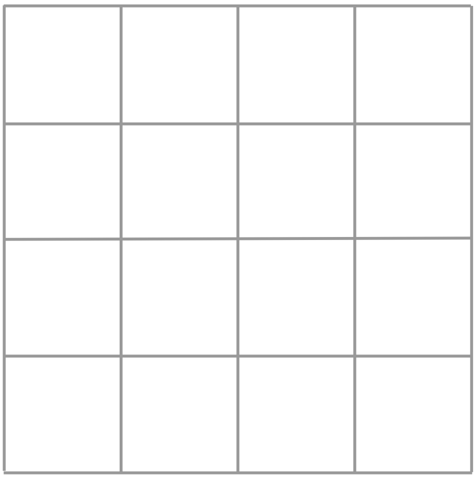

Figure 2: Sample computational grid to try different patterns.

(H*) Exercise 4: Flow Modeling

The incompressible Navier-Stokes equations for a two-dimensional flow with the vertical velocity component (y - direction) equal zero takes the form

$$\frac{\partial u}{\partial t} - C\frac{\partial^2 u}{\partial y^2} = g(t) \tag{6}$$

where u is the fluid velocity horizontal component (x - direction), $g(t)$ is the pressure gradient $-\partial p/\partial x$.

(a) Develop a numerical scheme to solve equation (6): use a first-order finite difference for the temporal derivative, a time-implicit, spatial second-order finite difference discretization for the spatial second-order derivative and a time-explicit evaluation of the function $g(t)$. You may further assume an equidistant discretization of the unit interval in space, $y \in [0, 1]$, using $N + 1$ grid points and a resulting mesh size $h := 1/N$. The time step shall be denoted by τ, the discrete velocity values for $u(t, y)$ by $u_i^n := u(n\tau, ih)$, $i = 0, ..., N$, and the discrete representation of the function $g(t)$ by $g^n := g(n\tau)$.

Formulate the relation at one particular time step $n \to n + 1$ in form of a linear system of equations

$$\sum_{j=1}^{N} A_{ij} u_j^{n+1} = b_i^n, \quad i = 0, ...N, \tag{7}$$

with matrix $(A_{ij}) \in \mathbb{R}^{N+1 \times N+1}$. The term b_i^n should contain all contributions from the previous time step n. Further assume homogeneous Dirichlet conditions at the outer boundaries, i.e. $u_0^{n+1} = 0$, $u_N^{n+1} = 0$. Give an exact definition of the matrix entries A_{ij} and right hand side entries b_i^n.

(b) Sketch the individual steps of the time stepping scheme to solve the finite difference formulation from (a) in pseudo-code.