Introduction to Scientific Computing II – Multigrid

Miriam Mehl, Michael Bader

Complete Fourier Analysis of the Two-Grid Iteration

We perform an analysis of the two-grid method for the 1D Poisson equation

$$u_{xx} = f$$

discretised with the help of the 3-point stencil

$$u_{xx}|_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

and with homogeneous boundary conditions, $u_0 = u_N = 0$.

In the following, we will examine the effect of the multigrid operators on error components with different frequencies

$$q_h^m = (\sin(m\pi hi))_{i=1,\dots,N-1}$$

(i.e., on the eigenmodes). We define high/low frequencies relative to the mesh size:

- $m < \frac{N}{2} \Rightarrow$ low frequency (representable on the coarse grid),
- $m \geq \frac{N}{2} \Rightarrow$ high frequency (not representable on the coarse grid).

Coarse-Grid Correction:

The fine-grid residual r^{old} is related to the error e^{old} via the residual equation: $r^{\text{old}} = Ae^{\text{old}}$. To this residual, we apply the following steps:

- restriction, with operator R, to the coarse grid
- solve the coarse-grid problem, which can formally be achieved by multiplying with the inverse of the coarse-grid Laplacian, A_H
- interpolation the resulting coarse-grid approximation to the fine grid, using an interpolation operator P.

The resulting correction is added to the solution, i.e., subtracted from the previous error:

$$e^{\text{new}} = e^{\text{old}} - PA_H^{-1}RA_h e^{\text{old}} = (E - PA_H^{-1}RA_h)e^{\text{old}}$$

with the identity matrix E.

Smoother:

Remember: for the fine grid operator

$$(A_h e)_i = \frac{1}{h^2}(e_{i+1} - 2e_i + e_{i-1}),$$

we know:

$$\begin{aligned} A_h q_h^m &= \dots \\ &= \frac{2(\cos(m\pi h) - 1)}{h^2} q_h^m. \end{aligned}$$

For the **damped Jacobi iteration** with $\omega = \frac{1}{2}$, we thus know about the reduction of the eigenmodes:

$$q_h^m \longrightarrow \frac{1}{2}(1 + \cos(m\pi h)) q_h^m.$$

Restriction by Injection:

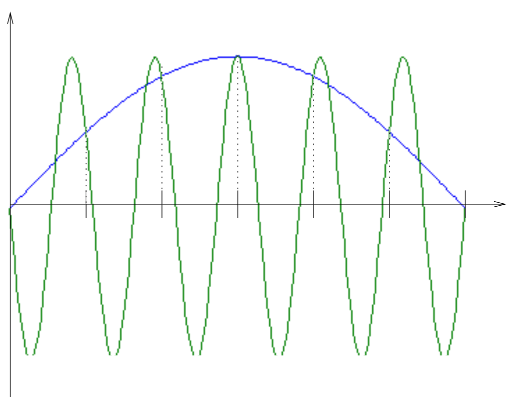
For all grid points x of the coarse grid ($x = m \cdot H$), we get (for injection):

$$\begin{aligned} Rq_h^m &= (\sin(m\pi hi))_{i=2,4,\dots,N-2} \\ &= (\sin(m\pi 2hi))_{i=1,\dots,\frac{N}{2}-1} = \begin{cases} q_{2h}^m & \text{if} \quad m < \frac{N}{2} \\ -q_{2h}^{N-m} & \text{if} \quad m \geq \frac{N}{2} \end{cases} \end{aligned}$$

Auxiliary calculation for $m \geq \frac{N}{2}$:

$$\begin{aligned} \sin(m\pi 2hi) &= \sin((m - N + N)\pi 2hi) \\ &= \sin((m-N)\pi 2hi) \underbrace{\cos(N\pi 2hi)}_{=\cos(2\pi i)=1} + \cos((m-N)\pi 2hi) \underbrace{\sin(N\pi 2hi)}_{=\sin(2\pi i)=0} \\ &= -\sin((N-m)\pi 2hi). \end{aligned}$$

Thus, injection-based restriction maps a pair m_1, m_2 of a (relative to the fine grid) low frequency $m_1 < \frac{N}{2}$ and a high frequency $m_2 = N - m_1 \geq \frac{N}{2}$ to the same coarse-grid frequency m_1 (see graph below for an illustration).



identity of sin with different frequencies on the coarse grid

Coarse-Grid Solution:

The coarse-grid solution step can be modelled via the inverse coarse grid operator (see fine grid operator)

$$A_H^{-1} q_H^m = \frac{H^2}{2(\cos(m\pi H) - 1)}.$$

Prolongation (via Linear Interpolation):

We write the linear interpolation as the following operator:

$$(Iq_H^m)_i = \begin{cases} \underbrace{(q_H^m)_{i/2}}_{=(q_h^m)_i} & \text{for} \quad i = 2, 4, \ldots, N-2 \\ \frac{1}{2}\big(\underbrace{(q_H^m)_{(i-1)/2}}_{=(q_h^m)_{i-1}} + \underbrace{(q_H^m)_{(i+1)/2}}_{=(q_h^m)_{i+1}}\big) & \text{for} \quad x = 1, 3, \ldots, N-1 \end{cases}$$

To eliminate the case distinction, we use that $\cos(\pi i) = \pm 1$ for even/odd i, and use the following auxiliary functions:

$$\begin{aligned} \frac{1}{2}(\cos(\pi i) + 1) &= \begin{cases} 1 & \text{for } i = 2, 4, \ldots, N-2 \\ 0 & \text{for } i = 1, 3, \ldots, N-1 \end{cases}, \\ \frac{1}{2}(-\cos(\pi i) + 1) &= \begin{cases} 0 & \text{for } i = 2, 4, \ldots, N-2 \\ 1 & \text{for } i = 1, 3, \ldots, N-1 \end{cases} \end{aligned}$$

With these functions, we eliminate the case distinction, and write:

$$\begin{aligned} (Iq_H^m)_i &= \frac{1}{2}(\cos(\pi i) + 1)(q_h^m)_i + \frac{1}{2}(-\cos(\pi i) + 1)\frac{1}{2}\left((q_h^m)_{i-1} + (q_h^m)_{i+1}\right) \\ &= \ldots = \frac{1}{2}(\cos(\pi i) + 1)(q_h^m)_i + \frac{1}{2}(-\cos(\pi i) + 1)\cos(m\pi h)(q_h^m)_i \\ &= \frac{1}{2}\left[(1 + \cos(m\pi h))(q_h^m)_i + (1 - \cos(m\pi h))\underbrace{\cos(\pi i)}_{=\cos(N\pi hi)}\underbrace{(q_h^m)_i}_{=\sin(m\pi hi)}\right] \end{aligned}$$

Note that, due to the factor $\cos(\pi i)$, this is not (yet) the desired multiple of $(q_h^m)_i$.
Using the following calculation

$$\underbrace{\sin((N-m)\pi hi)}_{=(q_h^{N-m})_i} = \underbrace{\sin(N\pi hi)}_{=0}\cos(m\pi hi) - \cos(N\pi hi)\underbrace{\sin(m\pi hi)}_{=(q_h^m)_i}.$$

we can "simplify" further and finally obtain:

$$(Iq_H^m)_i = \frac{1}{2}\big[(1 + \cos(m\pi h))(q_h^m)_i + (-1 + \cos(m\pi h))\sin((N-m)\pi hi)\big].$$

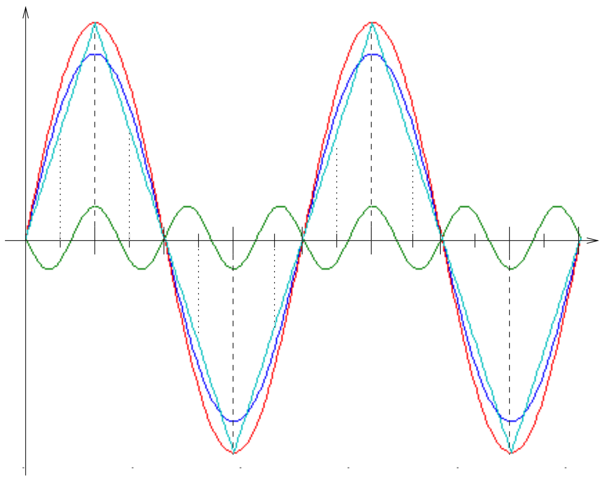
Thus, in vector notation, we get

$$Iq_H^m = \frac{1}{2}\big[(1 + \cos(m\pi h))q_h^m + (-1 + \cos(m\pi h))q_h^{N-m}\big].$$

This means that the interpolation produces new high-frequency components q_h^{N-m}.

⇒ **need for post-smoothing!**

The new high frequencies are introduced, as there is no exact replication of the functions ϕ by linear interpolation. At the new fine-grid points, the linear interpolant is no equal to the exact function value of ϕ. See the picture below for illustration.



coarse grid function to be interpolated (red)
interpolant (light blue)
low frequency component of the interpolant (dark blue)
high frequency component of the interpolant (green)

Composition of the Coarse-Grid Correction Operator $E - IA_H^{-1}RA_h$:

Step by step, we analyse the effect of the coarse-grid correction operator:

- computation of the residual:

$$A_h q_h^m = \frac{2(\cos(m\pi h) - 1)}{h^2} q_h^m$$

- restriction by injection:

$$RA_h q_h^m = \frac{2}{h^2}(\cos(m\pi h) - 1) \begin{cases} q_{2h}^m & \text{if} \quad m < \frac{N}{2} \\ -q_{2h}^{N-m} & \text{if} \quad m \geq \frac{N}{2} \end{cases}$$

- coarse-grid solution:

$$\begin{aligned} A_H^{-1} R A_h q_h^m &= \frac{2(\cos(m\pi h) - 1)}{h^2} \begin{cases} \dfrac{4h^2}{2(\cos(m\pi 2h) - 1)} q_{2h}^m & \text{if} \quad m < \frac{N}{2} \\ -\dfrac{4h^2}{2(\underbrace{\cos((N-m)\pi 2h)}_{=\cos(m\pi 2h)} - 1)} q_{2h}^{N-m} & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\ &= 4 \frac{\cos(m\pi h) - 1}{\cos(m\pi 2h) - 1} \begin{cases} q_{2h}^m & \text{if} \quad m < \frac{N}{2} \\ -q_{2h}^{N-m} & \text{if} \quad m \geq \frac{N}{2} \end{cases} \end{aligned}$$

- linear interpolation:

$$
\begin{aligned}
IA_H^{-1}RA_hq_h^m &= 4\frac{\cos(m\pi h)-1}{\cos(m\pi 2h)-1} \\
&\quad \begin{cases} \frac{1}{2}\left[(1+\cos(m\pi h))q_h^m + (-1+\cos(m\pi h))q_h^{N-m}\right] & \text{if} \quad m < \frac{N}{2} \\ -\frac{1}{2}\left[(1+\underbrace{\cos((N-m)\pi h)}_{=-\cos(m\pi h)})q_h^{N-m} + (-1+\underbrace{\cos((N-m)\pi h)}_{=-\cos(m\pi h)})q_h^m\right] & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\
&= 2\frac{\cos(m\pi h)-1}{\cos(m\pi 2h)-1}\begin{cases} (1+\cos(m\pi h))q_h^m + (-1+\cos(m\pi h))q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -(1-\cos(m\pi h))q_h^{N-m} - (-1-\cos(m\pi h))q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\
&= \frac{1}{\sin(m\pi h)^2}(1-\cos(m\pi h)) \\
&\quad \begin{cases} (1+\cos(m\pi h))q_h^m + (-1+\cos(m\pi h))q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -(1-\cos(m\pi h))q_h^{N-m} - (-1-\cos(m\pi h))q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\
&= \frac{1}{\sin(m\pi h)^2}\begin{cases} \sin(m\pi h)^2 q_h^m - (1-\cos(m\pi h))^2 q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -(1-\cos(m\pi h))^2 q_h^{N-m} + \sin(m\pi h)^2 q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\
&= \begin{cases} q_h^m - \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} + q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases}
\end{aligned}
$$

with the following auxiliary calculations:

$$
\begin{aligned}
\cos((N-m)\pi 2h) &= \underbrace{\cos(N\pi 2h)}_{=\cos(2\pi)=1}\cos(m\pi 2h) + \underbrace{\sin(N\pi 2h)}_{=\sin(2\pi)=0}\sin(m\pi 2h) \\
&= \cos(m\pi 2h), \\
\cos(N-m)\pi h &= \underbrace{\cos(N\pi h)}_{=\cos(\pi)=-1}\cos(m\pi h) + \underbrace{\sin(N\pi h)}_{=\sin(\pi)=0}\sin(m\pi h) \\
&= -\cos(m\pi h), \\
\cos(m\pi 2h) - 1 &= \cos(m\pi h)^2 - \sin(m\pi h)^2 - 1 = \\
&= \cos(m\pi h)^2 - \sin(m\pi h)^2 - \cos(m\pi h)^2 - \sin(m\pi h)^2 = -2\sin(m\pi h)^2.
\end{aligned}
$$

Thus, $IA_H^{-1}RA_hq_h^m$ corresponds to the following mapping:

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

where x_1, x_2 are coefficients in the linear subspace spanned by the basis $\left\{q_h^m, q_h^{N-m}\right\}$ with $m < \frac{N}{2}$.

- complete coarse-grid operator:
  With this, we get can represent the operator $E - IA_H^{-1}RA_hq_h^m$ with respect to the same linear subspaces as

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
$$

Thus, low frequency components are reduced to zero (zero entry first line, first column), but produce new high frequency errors (non-zero first line, second column. High frequency components are also reduced to zero (zero second line, second column) by the restriction as high frequencies cannot be displayed on the coarse grid, but are transformed to a low frequency error (non-zero second line, first column).

The corresponding two-dimensional matrix has the eigenvalues

$$\begin{aligned}\lambda_1 &= \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \text{ and} \\ \lambda_2 &= -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2}\end{aligned}$$

Two-Grid Operator:

The complete operator of the two-grid method consists of pre-smoothing, coarse-grid correction, and post-smoothing,and can thus be described by the matrix

$$\begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix}^n \begin{pmatrix} 0 & \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix}^m =$$

$$\begin{pmatrix} 0 & \frac{(1-\cos(m\pi h))^{2+m}(1+\cos(m\pi h))^n}{2^{m+n}\sin(m\pi h)^2} \\ \frac{(1-\cos(m\pi h))^{2+n}(1+\cos(m\pi h)^m}{2^{m+n}\sin(m\pi h)^2} & 0 \end{pmatrix}$$

Let's for example choose $m = n = 2$. Using

$$(1+\cos(m\pi h))(1-\cos(m\pi h)) = 1-\cos(m\pi h)^2 = \sin(m\pi h)^2,$$

we get the matrix

$$\begin{pmatrix} 0 & \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \\ \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} & 0 \end{pmatrix}$$

with the eigenvalues

$$\begin{aligned}\lambda_1 &= \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \text{ and} \\ \lambda_2 &= -\frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16}\end{aligned}$$

and eigenvectors

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} c \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} c$$

Compute the maximal eigenvalue:

$$\max_m |\lambda_1| = \max_m |\lambda_2| = \max_m \left| \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \right| \leq \frac{1}{16}.$$

Thus, the error will be reduced by a **factor of 16(!)** in each two-grid iteration. As this reduction factor **does not depend on N** (or h), the number of iterations required to achieve a certain accuracy will also be independent from the grid resolution N.

⇒ **Optimal convergence behaviour!**