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Scientific Computing II

Exercise 1

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Tutorial: Iterative Solvers

We have to solve the discretised one-dimensional Poisson equation on the unit interval with homgeneous Dirichlet boundary conditions:

$$\begin{aligned} \Delta u &= 0 \text{ in }]0,1[, \\ u &= 0 \text{ at } \partial]0,1[. \end{aligned}$$

The Laplacian is discretised on a regular grid by the 3-point-stencil

$$\Delta u(i \cdot h) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

We solve the resulting system of linear equations using

- Jacobi
- Damped Jacobi with $\omega = \frac{1}{2}$
- Gauss-Seidel
- Successive Overrelaxation (SOR) with the optimal overrelaxation factor $\omega = \frac{2}{1+\sin(\pi h)}$

We:

- Process the grid point by point.
- Change the value at the current point such that the local residual becomes zero or, in other words, the local equation is fulfilled.

- Repeatedly process all grid points until the solution is good enough.

Remark: The exact solution for the given system is $u = 0$ such that our current approximation at the same time represents the current error.

a) Establish the formula for the update of the value u_i at grid point i for all solvers.

b) Implement a Matlab programme performing one iteration with the given methods.

c) Use a uniform grid with meshsize $h = \frac{1}{4}$ and perform one sweep over all grid points using the following initial guesses for u:

1) $u_i = \sin(h\pi i)$ for all $i = 0, \ldots, 4$.

$$\Rightarrow \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \approx \begin{pmatrix} 0.000 \\ 0.707 \\ 1.000 \\ 0.707 \\ 0.000 \end{pmatrix}.$$

2) $u_i = \sin(3h\pi i)$ for all $i = 0, \ldots, 4$.

$$\Rightarrow \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \approx \begin{pmatrix} 0.000 \\ 0.707 \\ -1.000 \\ 0.707 \\ 0.000 \end{pmatrix}.$$

By what factor was the error (maximum norm) reduced? Sketch the initial guess together with the solution after this sweep.

d) Use a grid with meshsize $h = \frac{1}{8}$ and perform one sweep over all grid points using the following initial guesses for u:

1) $u_i = \sin(h\pi i)$ for all $i = 0, \ldots, 8$.

$$\Rightarrow \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} \approx \begin{pmatrix} 0.000 \\ 0.383 \\ 0.707 \\ 0.924 \\ 1.000 \\ 0.924 \\ 0.707 \\ 0.383 \\ 0.000 \end{pmatrix}.$$

2) $u_i = \sin(7h\pi i)$ for all $i = 0, \ldots, 8$.

$$\Rightarrow \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} \approx \begin{pmatrix} 0.000 \\ 0.383 \\ -0.707 \\ 0.924 \\ -1.000 \\ 0.924 \\ -0.707 \\ 0.383 \\ 0.000 \end{pmatrix}.$$

By what factor was the error (maximum norm) reduced? Sketch the initial guess together with the solution after this sweep.

e) What conclusions can you draw for the convergence of the method in dependence on the shape of the error?

f) Use Matlab to calculate the eigenvalues and eigenvectors for the iteration matrix for the Jacobi and the damped Jacobi method on the grid with meshsize $h = \frac{1}{8}$. Compare the values to the reduction obtained in d).