2. Multigrid

May 14, 2012

Multigrid - error frequency

Low-frequency errors on fine grids become high-frequency errors on coarse grids

- k-th eigen vector of iteration matrix r generated by
  $v_j^k = \sin\left(\frac{kj\pi}{n+1}\right) = \sin\left(k\pi \cdot j \cdot h\right)$
- high frequent: $k > n/2$
- Example: wave with $k = 4$ on two grids

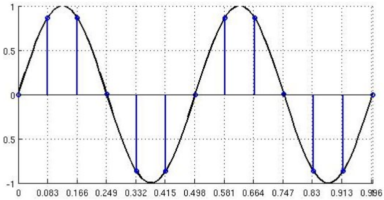


Figure: $v_j^4 = \sin\left(\frac{4 \cdot \pi \cdot j}{12}\right)$ on a grid with meshwidth $h = 1/12$.

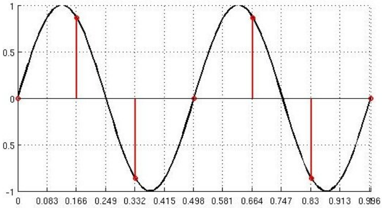


Figure: $v_j^4 = \sin\left(\frac{4 \cdot \pi \cdot j}{6}\right)$ on a grid with meshwidth $h = 1/6$.

Two-Grid - Correction Scheme

Two-Grid - Correction Scheme

- relax n_1 times on $A^h u^h = f^h$ on Ω^h with initial guess v^h
- compute fine-grid residual $r^h = f^h - A^h v^h$, restrict it to the coarse grid by $r^{2h} = R_h^{2h} r^h$
- solve / relax on $A^{2h} e^{2h} = r^{2h}$ on the coarse grid
- interpolate coarse grid error to the fine grid by $e^h = P_{2h}^h e^{2h}$
- correct fine grid solution by $v^h = v^h + e^h$
- relax n_2 times on $A^h u^h = f^h$ with corrected guess v^h.

Two-Grid - Example

Standard example: 1d Poisson-Equation

$$\begin{aligned} -u'' &= 0 \quad \text{in }]0;1[, \\ u &= 0 \quad \text{at } \partial]0;1[. \end{aligned}$$

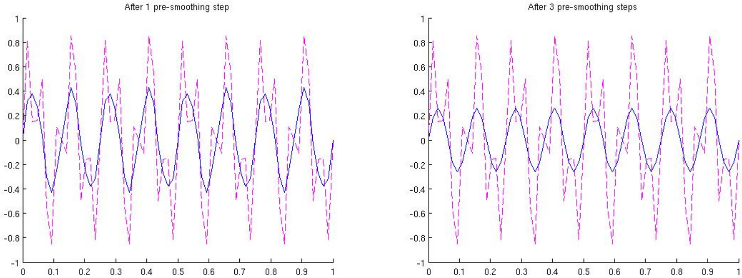
$h = \frac{1}{64}$ with initial guess

$$x_j^h = \frac{1}{2}\left[\sin\left(\frac{16j\pi}{n}\right) + \sin\left(\frac{40j\pi}{n}\right)\right]$$

We perform

- 3 Damped-Jacobi-iterations on the fine grid (presmoothing)
- 3 Damped-Jacobi-iterations on the coarse grid
- 3 Damped-Jacobi-iterations on the fine grid (postsmoothing)

Two-Grid - Example



After three smoothing steps, the high-frequency part of the error is smooth, the low-frequency part dominates.

Two-Grid - Example

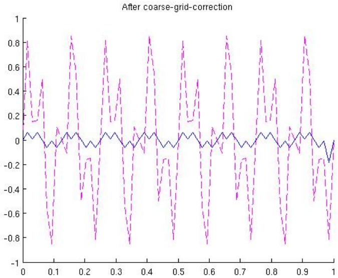


Figure: After the coarse-grid correction, the low-frequency part is reduced drastically, mainly high-frequencies remain.

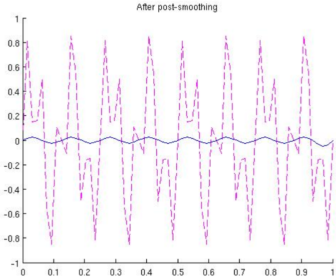


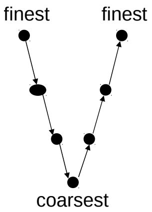
Figure: The high-frequency error is smoothed.

Multigrid - V-Cycle

Idea: Apply Correction-Scheme recursively, up to the coarsest grid level.

v^h **= procedure VCycle** $\left(v^h, f^h\right)$

1. Relax n_1 times on $A^h u^h = f^h$ with given initial guess v^h.
2. If $\Omega^h =$ coarsest grid then goto step 4.
   Else
   $f^{2h} = R_h^{2h} \cdot res^h,$
   $v^{2h} = 0,$
   $v^{2h} = VCycle\left(v^{2h}, f^{2h}\right).$
3. Correct $v^h = v^h + P_{2h}^h v^{2h}$
4. Relax n_2 times on $A^h u^h = f^h$ with corrected initial guess v^h.



V-cycle